Rank rigidity of CAT(0) groups

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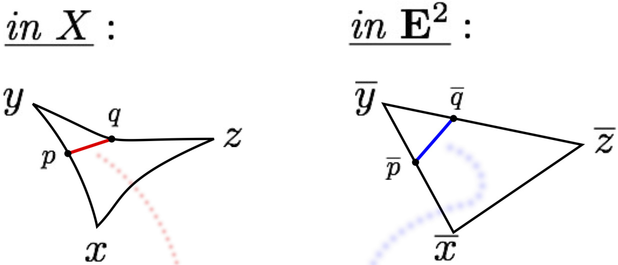
The CAT(0) inequality

A complete geodesic metric space (X, d) is CAT(0), if all geodesic triangles $\triangle xyz$ are thinner than their Euclidean comparison triangles:



$$\boxed{d(p, q)} \leq \boxed{d_2(\bar{p}, \bar{q})}$$

CAT(0) – Why should I care?

Properties:

- Convexity of the metric

  $\rightsquigarrow$ convex analysis can be done here!

- Uniqueness of geodesics
- Nearest point projections to closed convex subspaces
- Contractibility
- Universal cover of locally CAT(0) is CAT(0), e.g.:
  - a 2-dim'l square complex with vertex links of girth$\geq$ 4;
  - more generally a cubical complex all whose links are flags.

  $\rightsquigarrow$ a starategy for constructing a $K(G,1)$

- A compact set has a circumcenter

  $\rightsquigarrow$ have a grip on compact subgroups of $\mathrm{Isom}\,(X)$

CAT(0) Groups

By a CAT(0) group we mean a group G, together with a proper, co-compact isometric (*geometric*) action on a CAT(0) space X.

Properties:

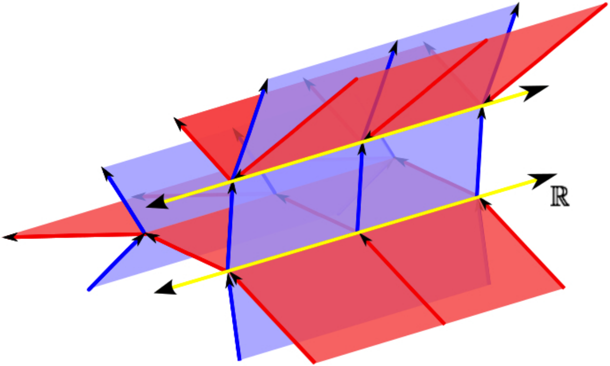
- A finite order element fixes a point in X (*elliptic*);
- An element of infinite order acts as a translation on a geodesic line in X (*hyperbolic*);
- There are only finitely many elliptic conjugacy classes;
- No infinite subgroup of G is purely elliptic. (Swenson)

Theorem (Flat Torus theorem)

If $H < G$ is a free abelian subgroup of rank d, then H stabilizes an isometrically embedded d-flat $F \subset X$, on which it acts co-compactly by translations.

Examples

$\langle a, b \mid - \rangle \times \mathbb{Z}$ acts geometrically on the product of a 4-regular tree with the real line:



Examples

Other examples:

- Lattice in $\mathrm{Isom}(\mathbb{H}^n)$: becomes a CAT(0) group upon excising a maximal disjoint family of precisely invariant horoballs;
- Coxeter groups acting on Davis-Moussong complexes;
- Direct products of free groups;
- More generally, right-angled Artin groups;
- Fundamental groups of piecewise-NPC complexes with large links.

Visual Boundary

Two geodesic rays $\gamma, \gamma' : [0, \infty) \to X$ are *asymptotic*, if $d(\gamma(t), \gamma'(t))$ is bounded.

The visual boundary ∂X of a CAT(0) space X is the set of asymptoticity classes of geodesic rays in X.

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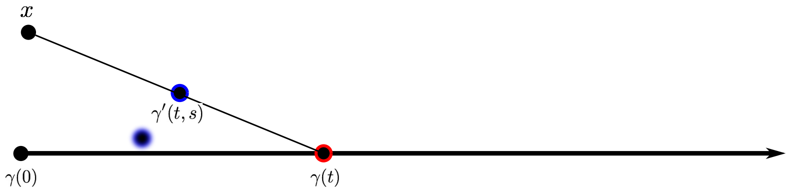

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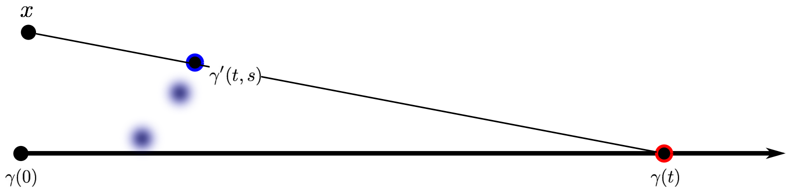

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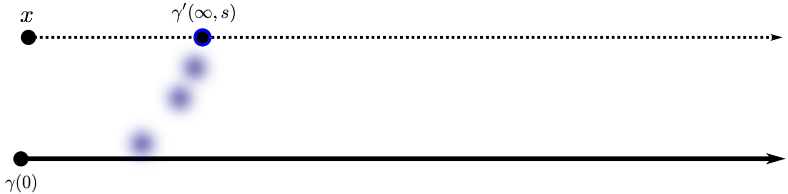

Theorem

Let $x \in X$. Then every asymptoticity class in ∂X contains a unique representative emanating from x.

Why boundaries?

Boundaries were born to answer coarse geometric questions, e.g.:

- *Which subgroups of G stabilize large-scale features?*

  …for example, extend the idea (and role) of parabolic subgroups encountered in the classical groups; radial vs. tangential convergence.

- *If X is so-and-so, what is G?*

  …Mostow rigidity utilizes the conformal structure on the ideal sphere for classifying G up to conjugacy according to the homotopy type of $G\backslash\mathbb{H}^3$.

Cone boundary $\partial_\infty X$ *vs.* Tits boundary $\partial_T X$

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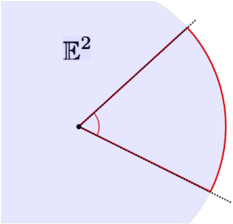

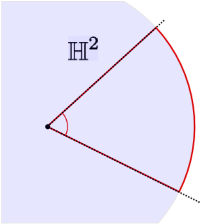

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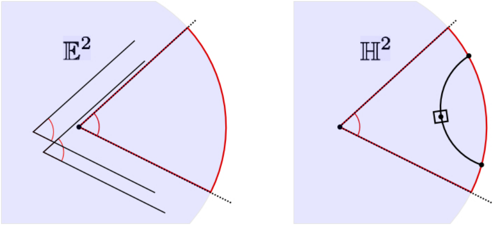
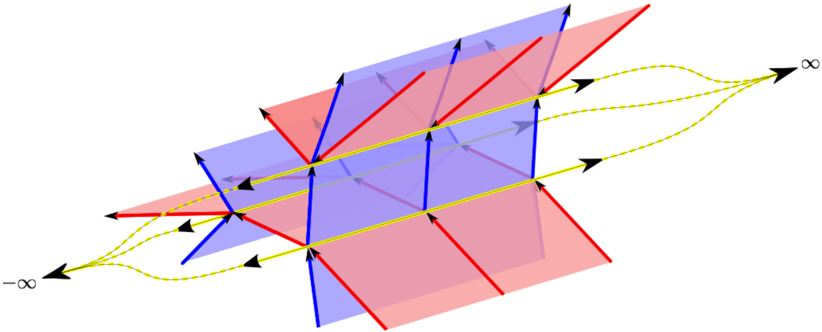


- …I can stare at both of them at the same time, no matter where I stand. (*Tits metric*, $\partial_T X$ – use angles)

Cone boundary $\partial_\infty X$ *vs.* Tits boundary $\partial_{\mathrm{T}} X$

Example: $G = \mathbf{F}_2 \times \mathbb{Z} \curvearrowright \mathrm{T}_4 \times \mathbb{R}$



$$\partial\left(\mathrm{T}_4 \times \mathbb{R}\right) = \partial \mathrm{T}_4 * \partial \mathbb{R} = \{\text{Cantor set}\} * \{\pm\infty\}$$

$\partial_\infty X$ is coarser than $\partial_{\mathrm{T}} X$!!

Cone boundary $\partial_\infty X$ *vs.* Tits boundary $\partial_T X$

More generally, we have good news:

- $\partial_T X$ is a complete CAT(1) space (Kleiner-Leeb)
- $\partial(X \times Y) = \partial X * \partial Y$ for *both* the Cone and Tits boundaries (Berestovskij)
- The Tits metric is lower semi-continuous on $\partial_\infty X \times \partial_\infty X$.
- The following are equivalent: (Gromov?)
  - G is Gromov-hyperbolic,
  - $\partial_T X$ is discrete,
  - X contains no 2-flat.

The Bad News

- $\partial_T X$ is not locally compact ($F_2 \times \mathbb{Z}$)
- G one-ended but $\partial_T X$ not connected (Croke-Kleiner)
- G determines neither $\partial_\infty X$ nor $\partial_T X$ (Croke-Kleiner)
- Many join-irreducible examples of $\partial_\infty X$ not locally connected (Mihalik-Ruane)
- $\partial_\infty X$ not 1-connected though G is 1-connected at infinity (Mihalik-Tschantz)
- If there is a round $\mathbb{S}^d \subset \partial_T X$, is there a periodic $\mathbb{E}^{d+1} \subset X$? (Gromov, Wise)
- $\partial_T X$ is connected iff $\mathrm{diam}\partial_T X \leq \frac{3\pi}{2}$. (Ballmann-Buyalo, Swenson-Papasoglu)

  What does it mean for $\partial_T X$ to have diameter$\leq \pi$?

Rank One

Let X be a proper CAT(0) space.

Rank one

A rank one geodesic in X is a geodesic line not bounding a flat half-plane. A rank one isometry is a hyperbolic isometry $g \in \mathrm{Isom}\,(X)$ having a rank one axis. A group $G < \mathrm{Isom}\,(X)$ has rank one if it contains a rank one isometry (otherwise G has higher rank).

- **Origin:** rank one Lie groups and discrete subgroups thereof;
- **More generally:** hyperbolic and relatively-hyperbolic groups;
- **Typical behaviour:** Convergence dynamics, mimicking compactness properties of univalent analytic mappings.

Rank One and Convergence Dynamics

Discrete Convergence (Gehring-Martin)

Every infinite $F \subset G$ contains a sequence g_n converging on $\partial_\infty X$ to a constant map uniformly on compacts in $\partial_\infty X \smallsetminus \{pt\}$.

Morally (or loosely) speaking,

G of higher rank $\Leftrightarrow$ flats abound in X $\Rightarrow$ DCG fails

How? – e.g., if $g \in G$ has an axis bounding a flat half-plane F, then g^n cannot collapse ∂F to $g(\infty)$.

Some properties of rank one groups:

- Many non-abelian free subgroups
- More 'interesting' boundaries
- Better chances for splittings over 'nice' subgroups

Compressiblity

To study higher rank groups, we introduce:

Compressible pairs (Swenson-G.)

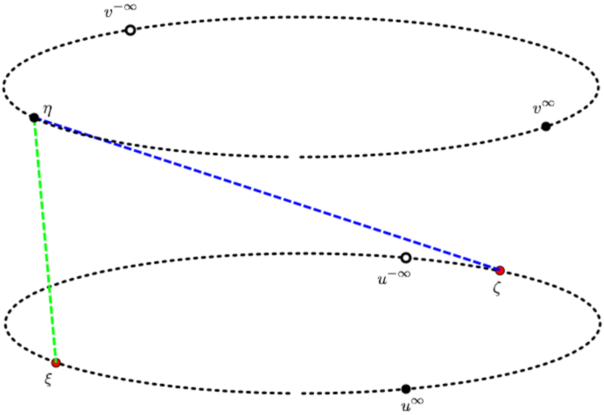
A pair $p, q \in \partial X$ is *G-compressible* if there are $g_n \in G$ such that $g_n p \to p_\infty$, $g_n q \to q_\infty$ but $\mathrm{d}_{\mathrm{T}}(p_\infty, q_\infty) < \mathrm{d}_{\mathrm{T}}(p, q)$.
$A \subset \partial X$ is *incompressible* if contains no compressible pair.

Examples:

- Rank one $\Rightarrow$ No non-degenerate incompressible sets
- $X = \mathbb{E}^m \Rightarrow$ entire boundary is G-incompressible
- More generally, $\partial_{\mathrm{T}} X$ compact $\Rightarrow \partial_{\mathrm{T}} X$ is G-incompressible

Example: $G = \mathrm{F}_2 \times \mathrm{F}_2$ is compressible

Action of $g^n = u^n \times v^n$ with $1 \neq u, v \in \mathrm{F}_2$ causes massive compressions away from the repelling points (ξ, η arbitrary):



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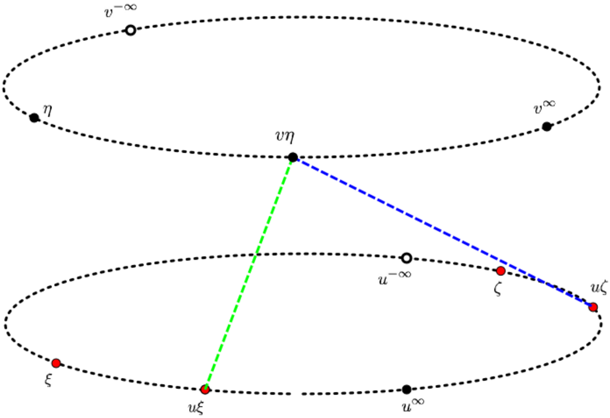

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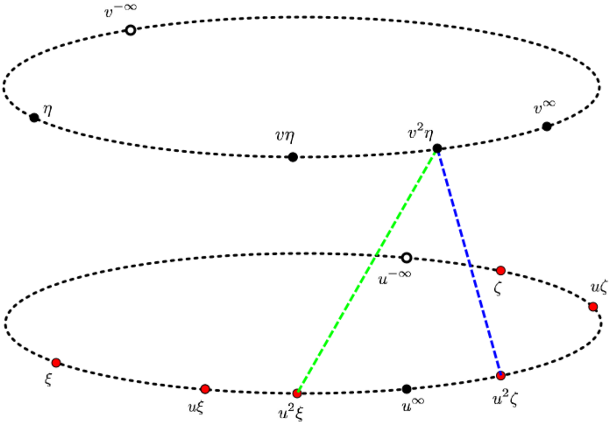

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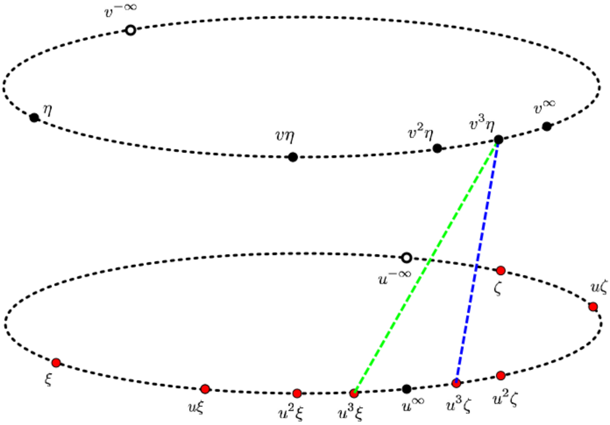

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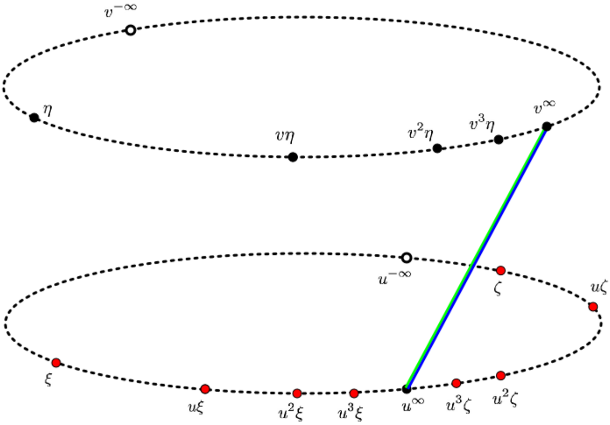

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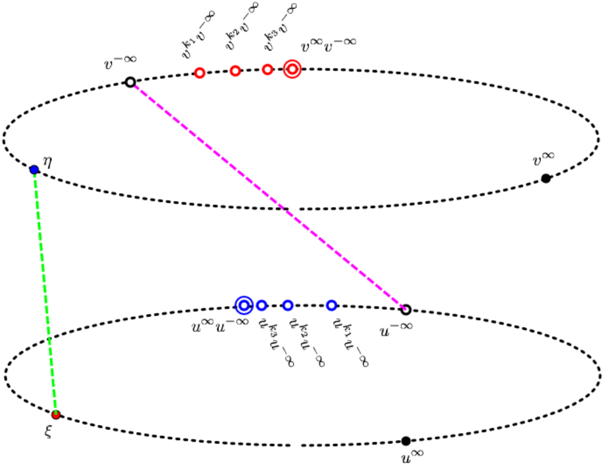

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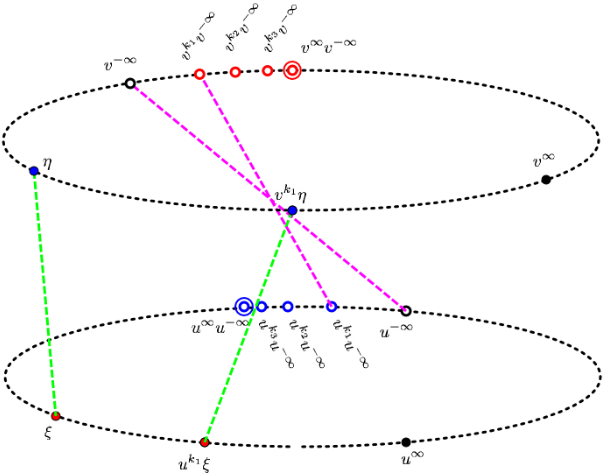

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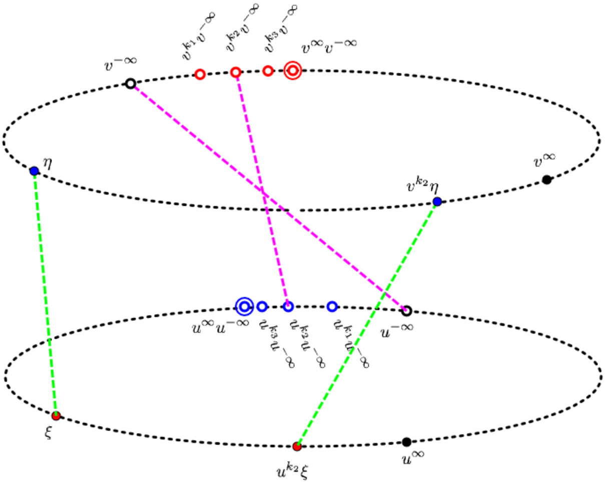

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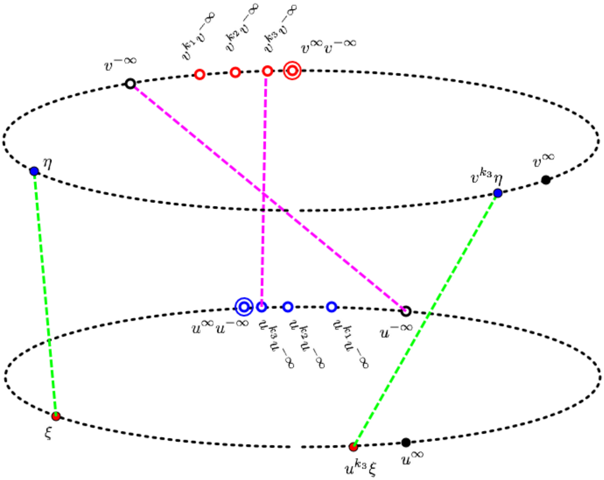

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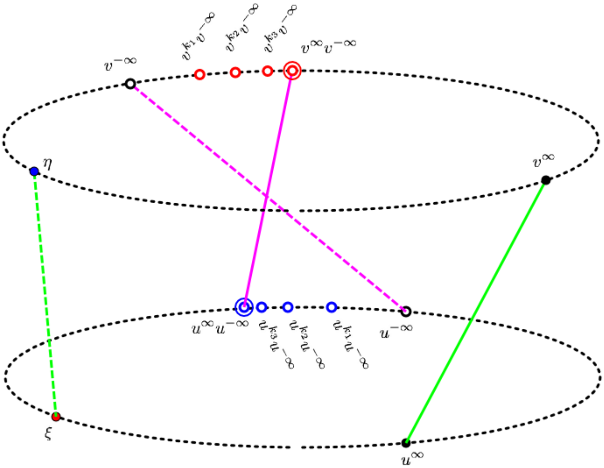

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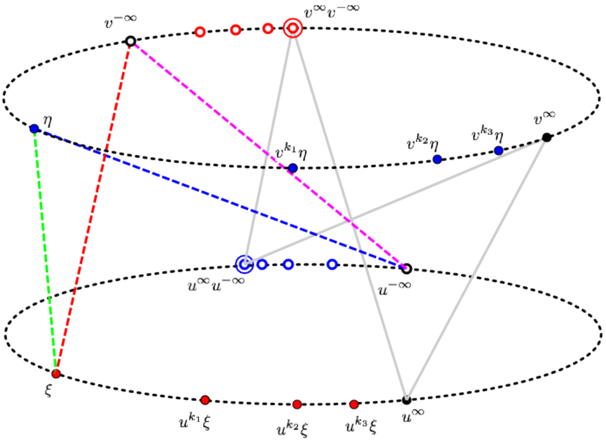

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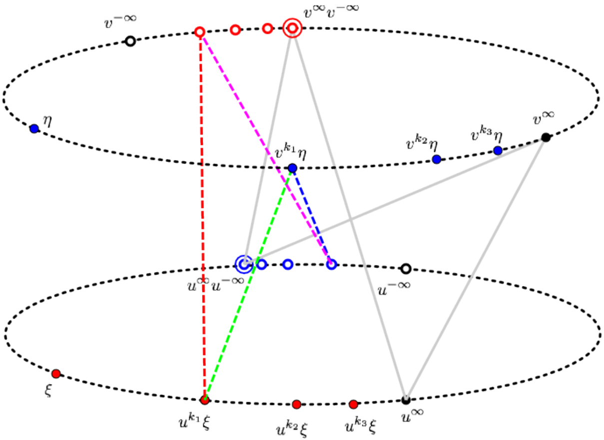

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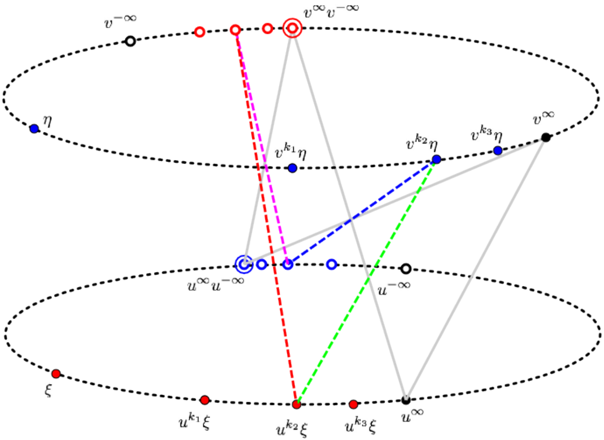

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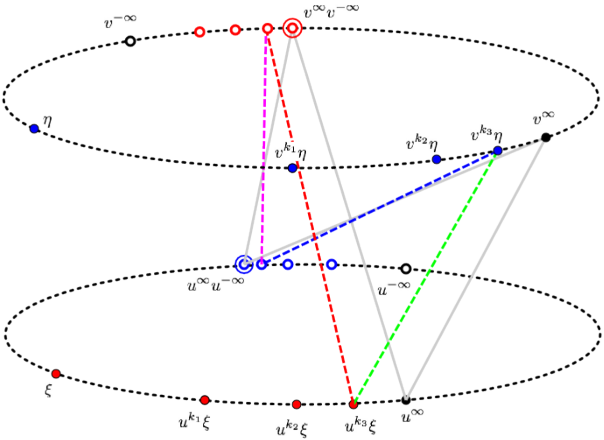

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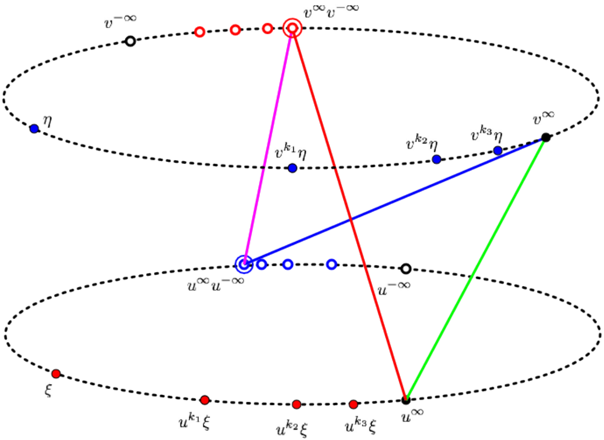

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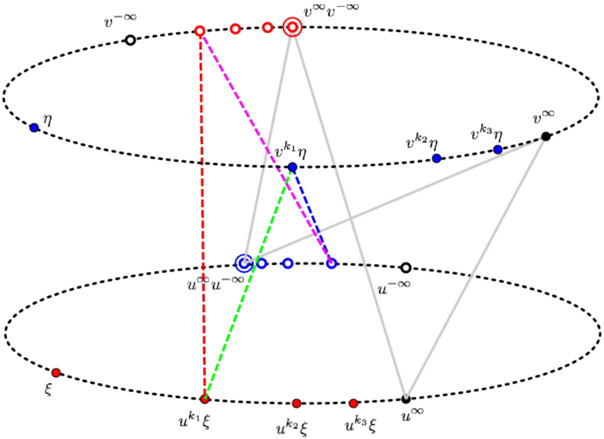
Question: How much of this can be retained *without* prior knowledge about group/space structure?



Goal: the Rank-Rigidity Conjectures

Suppose $G \curvearrowright X$ is a CAT(0) group.

Conjecture: Closing Lemma (Ballman-Buyalo)

If $\mathrm{diam}\,\partial_{\mathrm{T}} X > \pi$, then G has rank one.

The best known bound is $\frac{3\pi}{2}$, due to Swenson and Papasoglu.

Conjecture: Rank-rigidity (Ballman-Buyalo)

If $\mathrm{diam}\,\partial_{\mathrm{T}} X = \pi$ and X is irreducible, then X is either a symmetric space or a Euclidean building.

Known for:

- Riemannian manifolds (Ballman)
- Cell complexes of low dimensions (Ballman and Brin)
- Cubings (Caprace and Sageev)

A sample of our results

Theorem (G.-Swenson)

Let $G \curvearrowright X$ be a CAT(0) group of higher rank, and let d denote the geometric dimension of $\partial_{\scriptscriptstyle T} X$. Then

$$\mathrm{diam}\partial_{\scriptscriptstyle T} X \leq 2\pi - \arccos\left(-\frac{1}{d+1}\right) .$$

Theorem (G.-Swenson)

Let $G \curvearrowright X$ be a CAT(0) group of higher rank. TFAE:

1. G is virtually-Abelian;
2. X contains a virtually G-invariant coarsely dense flat;
3. G stabilizes a non-degenerate maximal incompressible subset of ∂X.

Approach through the boundary $\partial_{\mathrm{T}} X$

Most promising results in the direction of rank rigidity are:

- Leeb: If X is geodesically-complete and $\partial_{\mathrm{T}} X$ is a join-irreducible spherical building then X is a symmetric space or Euclidean building;
- Lytchak: If $\partial_{\mathrm{T}} X$ is geodesically-complete and contains a proper closed subspace closed under taking antipodes, then $\partial_{\mathrm{T}} X$ is a spherical building.

Question: Assume G has higher rank. How to use $G \curvearrowright X$ for obtaining a classification of its possible boundaries?

Some known structural results

Let Z be a finite-dimensional complete CAT(1) space:

- Lytchak: If Z is geodesically-complete then

$$Z = \underbrace{\mathbb{S}^n}_{\textit{sphere}} * \underbrace{Z_1 * \cdots * Z_k}_{\text{irred. buildings}} * \underbrace{Y_1 * \cdots * Y_l}_{\text{irred. none of the above}}$$

  This decomposition is unique.

- Swenson: There always is a decomposition

$$Z = \underbrace{\mathbb{S}(Z)}_{\text{sphere}} * \underbrace{E(Z)}_{\text{no sphere factor}}$$

  Moreover, $\mathbb{S}(Z)$ is the set of suspension points of Z *and the decomposition is unique.*

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π-Convergence (Swenson-Papasoglu)

For every $\epsilon \in [0, \pi]$, every infinite $F \subset G$ contains a sequence g_n converging on $\partial_\infty X$ into a Tits ball $\mathrm{B}_{\mathrm{T}}(p, \epsilon)$ uniformly on compacts in $\partial_\infty X \smallsetminus \mathrm{B}_{\mathrm{T}}(n, \pi - \epsilon)$.

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Special attention on the words 'converging' and 'into':

1. No actual limiting map $\partial_\infty X \smallsetminus \mathrm{B}_{\mathrm{T}}(n, \pi - \epsilon) \to \mathrm{B}_{\mathrm{T}}(p, \epsilon)$.
2. Varying limits can be constructed, but are –
   - choice-dependent, and
   - restricted to Tits-compact sets.

Our contribution: use the Ellis semi-group

Instead of limits of the form $\lim_{n\to\infty} g_n p$, work with G-ultra-limits $\omega p = \lim_{g\to\omega} gp$, computed in the cone compactification $\widehat{X}$ of X.

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- The inversion map extends to a continuous involution $\omega \mapsto S\omega$ (*antipode*).
- **Caveat:**
  1. $\omega \curvearrowright \partial_\infty X$ need not be continuous,
  2. $S(\omega\nu) \neq (S\nu)(S\omega)$.

Ultra-filter Miracles

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Most importantly: Can appeal to compactness of βG.

Our main tool: Folding and Total Folding

Folding Lemma (Swenson-G.)

Let d be the geometric dimension of $\partial_{\mathrm{T}} X$. Then for every $(d+1)$-flat F there exist $\omega_0 \in \beta G$ and a $(d+1)$-flat F_0 such that

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Corollaries: For $\mathbb{S}_0$ as above,

- every minimal closed invariant subset of $\partial_\infty X$ intersects $\mathbb{S}_0$;
- $\mathbb{S}_0$ contains an isometric copy of any incompressible subset $A \subset \partial X$;
- Every maximal incompresible subset of ∂X is isometric to a compact π-convex subset of a round sphere.

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Total Folding (Swenson-G.)

There exists $\nu_0 \in \beta G$ such that

- $\nu_0 \partial X$ is a maximal incompressible subset of maximal volume (MVI),
- $\nu_0^2 = \nu_0$ in βG, and $\nu_0 \partial X \subset \mathbb{S}_0$.

Moreover, any two MVI's are isometric and (geometrically) interiorly-disjoint.

Remarks:

- Rank one implies ∂X is compressible. Converse?
- Does higher rank imply $\mathbb{S}_0$ is covered by incompressibles?

  $\rightsquigarrow$ A positive answer implies $\mathrm{diam} \partial_{\scriptscriptstyle T} X = \pi$
- Is ∂X covered by MVI's?

  $\rightsquigarrow$ Through Lytchak, this would imply rank rigidity!

Endspiel: let's try to prove something

Theorem (G.-Swenson)

Let $G \curvearrowright X$ be a CAT(0) group of higher rank. TFAE:

1. G is virtually-Abelian;
2. X contains a virtually G-invariant coarsely dense flat;
3. G stabilizes a non-degenerate maximal incompressible subset of ∂X.

The plan to prove $(3) \Rightarrow (1)$**:**

- Prove that $\partial_{\mathrm{T}} X$ and $\partial_\infty X$ coincide with the round sphere;
- Hit this on the head with Shalom's QI characterization of virtually-Abelian groups.

But first we need to find a candidate sphere living inside ∂X.

The sphere of poles

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- In fact, A is the spherical join of $\mathbb{P}$ with a compact convex spherical polytope.

Characterizing virtually-Abelian groups

Main direction: if G stabilizes a max incompressible set A, then G is virtually-Abelian.

- Write $A = \mathbb{P} * B$ using Swenson's decomposition, and prove B must be empty (This is the main step where group theory is involved).

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THE END, THANK YOU!

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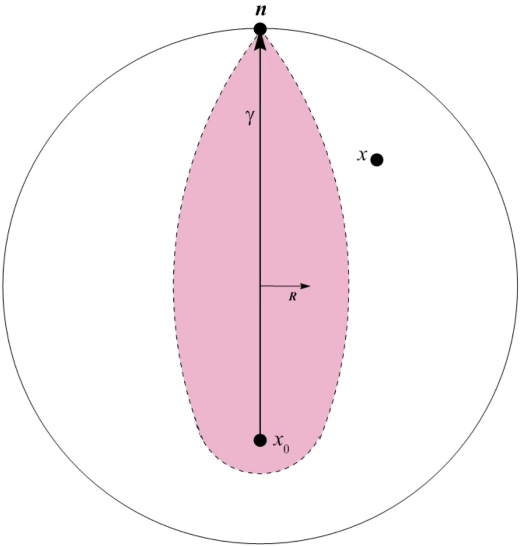

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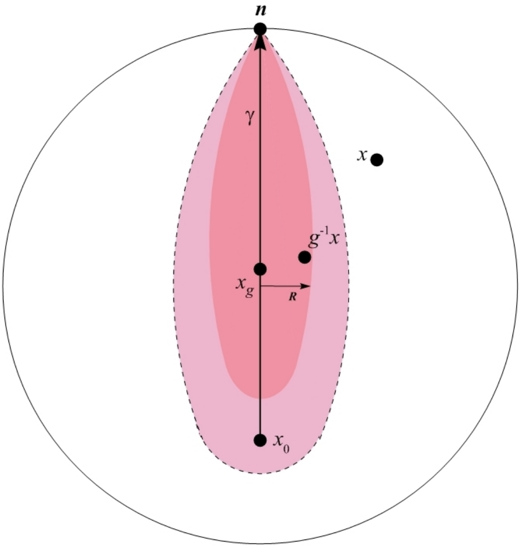

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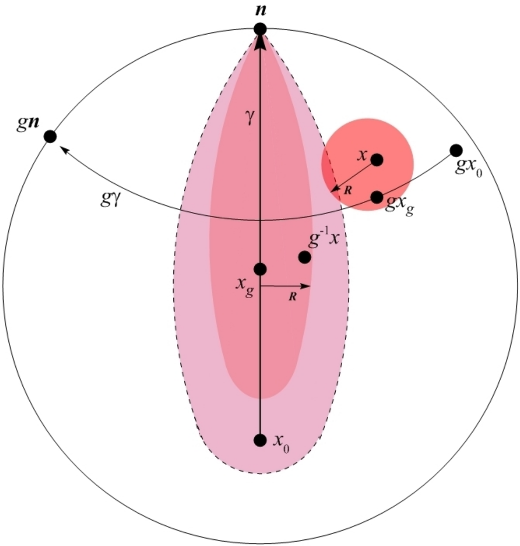

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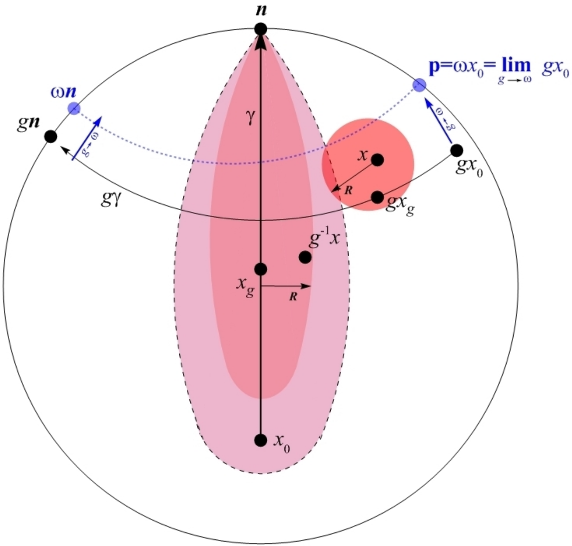

What Makes Folding Work?

Properties of pulling (Swenson-G.)

Suppose $\omega \in \beta G$ pulls from a point $n \in \partial X$. Then:

1. if F is a flat with $n \in \partial F$, then ω maps ∂F isometrically onto the boundary of a flat;
2. if $\mathrm{d}_{\mathrm{T}}(n, a) \leq \pi$, then ω restricts to an isometry on $[n, a]$;
3. if $\mathrm{d}_{\mathrm{T}}(n, a) \geq \pi$, then $\omega a = \omega(\infty)$. Thus,
4. ω maps ∂X into the geodesic suspension of ωn and $\omega(\infty)$, preserving boundaries of flats through n.

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