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On the Optimality of Napoleon Triangles

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Abstract An elementary geometric construction, known as Napoleon's theorem, produces an equilateral triangle, obtained from equilateral triangles erected on the sides of any initial triangle: the centres of the three equilateral triangles erected on the sides of the arbitrarily given original triangle, all outward or all inward, are the vertices of the new equilateral triangle. In this note, we observe that two Napoleon iterations yield triangles with useful optimality properties. Two inner transformations result in a (degenerate) triangle, whose vertices coincide at the original centroid. Two outer transformations yield an equilateral triangle, whose vertices are closest to the original in the sense of minimizing the sum of the three squared distances.

Keywords Napoleon Triangle · Optimality · Torricelli Configuration · Fermat Problem · Torricelli Point

Mathematics Subject Classification (2000) 51-XX · 90-XX

1 Introduction

In elementary geometry, one way of constructing an equilateral triangle from any given triangle is as follows: in a plane the centres of equilateral triangles erected, either all externally or all internally, on the sides of the given triangle form an equilateral triangle (see [1, Chapter 3.3] and the survey [2]). This result is generally referred to as *Napoleon's theorem*, notwithstanding its dubious

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origins — see [2] and [3] for a detailed history of the theorem. We will refer to these constructions as *the outer and inner Napoleon transformations*, and the associated equilateral triangles as *the outer and inner Napoleon triangles* of the original triangle, respectively. Conversely, given its outer and inner Napoleon triangles in position (i.e., they are oppositely oriented and have the same centroid), the original triangle is uniquely determined; for this and related results we refer to [4] and [5]. In other words, the converse of Napoleon's theorem offers a parametrization of a triangle in terms of equilateral triangles. A fascinating application of Napoleon triangles is the planar tessellation used by Escher: a plane can be tiled using congruent copies of the hexagon, defined by the vertices of any triangle and its uniquely paired outer Napoleon triangle, known as *Escher's theorem* [6].

Equilaterals built on the sides of a triangle make a variety of appearances in the classical literature. Torricelli uses this construction to solve Fermat's problem: locate a point minimizing the sum of distances to the vertices of a given triangle — one of the first problems of location science [7]. The unique solution of this problem is known as *the Torricelli point* of the given triangle, located as follows [8]. If an internal angle of the triangle is greater than 120°, then the Torricelli point is at that obtuse vertex. Otherwise, the three lines joining opposite vertices of the original triangle and externally erected triangles are concurrent, and they intersect at the Torricelli point. The figure, defined by the original triangle and the erected equilateral triangles, is referred to as *the Torricelli configuration* (see [9,10]), and the new vertices of this figure form the so-called *vertex set of the Torricelli configuration.* It also bears mentioning that explicit solutions in nonlinear optimization are very rare. The Fermat problem for three points is such a special case, and its generalization to more points has no explicit solution [7].

In this paper, we demonstrate some remarkable, but not immediately obvious, optimality properties of twice iterated Napoleon triangles. First, two composed inner Napoleon transformations of a triangle collapse the original one to a point located at its centroid which, by definition, minimizes the sum of squared distances to the vertices of the given triangle. Surprisingly, two composed outer Napoleon transformations yield an equilateral triangle, optimally aligned with the original triangle by virtue of minimizing the sum of squared distances between the paired vertices (Theorem 3.1). It is important to emphasize that this is another rare instance of a nonlinear optimization problem that admits an explicit solution.

2 Torricelli and Napoleon Transformations

For any ordered triple $\mathbf{x} = [\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]^{\mathrm{T}} \in \mathbb{R}^{3d}$ of vectors of the d-dimensional Euclidean space $\mathbb{R}^d$, let $\mathbf{R}_{\mathbf{x}}$ denote the *rotation matrix* corresponding to a counter-clockwise rotation by $\pi/2$ in the plane, defined by orthonormal vectors n and t, in which the triangle $\triangle_{\mathbf{x}}$ formed by $\mathbf{x}$ is positively oriented (i.e., its vertices in counter-clockwise order follow the sequence $\ldots \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow \ldots$),

$$\mathbf{R}_{\mathbf{x}} := [\,\mathrm{n},\ \mathrm{t}\,] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [\,\mathrm{n},\ \mathrm{t}\,]^{\mathrm{T}}, \tag{1}$$

where

$$\mathrm{n} := \begin{cases} \frac{\mathrm{x}_2 - \mathrm{x}_1}{\|\mathrm{x}_2 - \mathrm{x}_1\|}, & \text{if } \mathrm{x}_1 \neq \mathrm{x}_2, \\ \frac{\mathrm{x}_3 - \mathrm{x}_2}{\|\mathrm{x}_3 - \mathrm{x}_2\|}, & \text{otherwise}, \end{cases} \quad \mathrm{t} := \begin{cases} \in \left\{ \mathrm{z} \in \mathbb{S}^{d-1} \middle| \mathrm{n}^{\mathrm{T}}\mathrm{z} = 0 \right\}, & \text{if } \mathbf{x} \text{ is collinear}, \\ \mathbf{P}(\mathrm{n}) \frac{\mathrm{x}_3 - \mathrm{x}_1}{\|\mathrm{x}_3 - \mathrm{x}_1\|} & , \text{otherwise}. \end{cases} \tag{2}$$

Here, $\|.\|$ denotes the standard Euclidean norm on $\mathbb{R}^d$, and $\mathbf{P}(\mathrm{n}) := \mathbf{I}_d - \mathrm{n}\mathrm{n}^{\mathrm{T}}$ is the projection onto $T_{\mathrm{n}}\mathbb{S}^{d-1}$ (the tangent space of the $(d-1)$-sphere $\mathbb{S}^{d-1}$ at point $\mathrm{n} \in \mathbb{S}^{d-1}$), and $\mathbf{I}_d$ is the $d \times d$ identity matrix, and $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of a matrix $\mathbf{A}$. For any trivial triangle $\triangle_{\mathbf{x}}$, all of whose vertices are located at the same point, we fix $\mathbf{R}_{\mathbf{x}} = \mathbf{0}$ by setting $\frac{\mathrm{x}}{\|\mathrm{x}\|} = 0$ whenever $\mathrm{x} = 0$. Note that $\triangle_{\mathbf{x}}$ is both positively and negatively oriented if $\mathbf{x}$ is collinear. Consequently, to define a plane containing such $\mathbf{x}$, we select an arbitrary vector t perpendicular to n in (2). It is also convenient to denote by $\mathrm{c}(\mathbf{x})$ the *centroid* of $\triangle_{\mathbf{x}}$, i.e., $\mathrm{c}(\mathbf{x}) := \frac{1}{3}\sum_{i=1}^{3} \mathrm{x}_i$.

In general, the Torricelli and Napoleon transformations of three points in Euclidean d-space can be defined based on their original planar definitions in a 2-dimensional subspace of $\mathbb{R}^d$ containing $\mathbf{x}$. That is to say, for any $\mathbf{x} \in \mathbb{R}^{3d}$, select a 2-dimensional subspace of $\mathbb{R}^d$ containing $\mathbf{x}$, and then construct the erected triangles on the side of $\triangle_{\mathbf{x}}$ in this subspace to obtain the Torricelli and Napoleon transformations of $\mathbf{x}$, as illustrated in Figure 1. Accordingly, let $\mathrm{T}_{\pm} : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ and $\mathrm{N}_{\pm} : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ denote the *Torricelli and Napoleon*

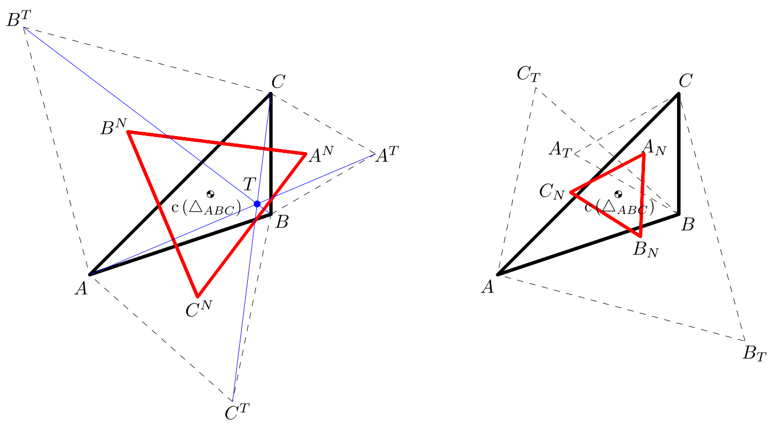

Fig. 1: An illustration of (left) the Torricelli point T, the outer Torricelli configuration with $\triangle_{A^T B^T C^T}$ and the outer Napoleon triangle $\triangle_{A^N B^N C^N}$, and (right) the inner Torricelli configuration with $\triangle_{A_T B_T C_T}$ and the inner Napoleon triangle $\triangle_{A_N B_N C_N}$ of a triangle $\triangle_{ABC}$. Note that centroids of the vertices of Torricelli configurations, Napoleon triangles and the original triangle all coincide, i.e., $\mathrm{c}(\triangle_{ABC}) = \mathrm{c}(\triangle_{A^T B^T C^T}) = \mathrm{c}(\triangle_{A^N B^N C^N}) = \mathrm{c}(\triangle_{A_T B_T C_T}) = \mathrm{c}(\triangle_{A_N B_N C_N})$.

transformations where the sign, $+$ and $-$, determines the type of the transformation, inner and outer, respectively. Denoting by $\otimes$ the Kronecker product [11], one can write closed form expressions of the Torricelli and Napoleon transformations as follows.

Lemma 2.1 *The Torricelli and Napoleon transformations of any triple* $\mathbf{x} \in \mathbb{R}^{3d}$ *on a plane containing* $\mathbf{x}$ *are, respectively, given by*

$$\mathrm{T}_{\pm}(\mathbf{x}) = \left(\frac{1}{2}\mathbf{K} \pm \frac{\sqrt{3}}{2}(\mathbf{I}_3 \otimes \mathbf{R}_{\mathbf{x}})\,\mathbf{L}\right)\mathbf{x}, \tag{3}$$

$$\mathrm{N}_{\pm}(\mathbf{x}) = \frac{1}{3}\big(\mathbf{K}\mathbf{x} + \mathrm{T}_{\pm}(\mathbf{x})\big)\,, \tag{4}$$

where

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \otimes \mathbf{I}_d \quad \textit{and} \quad \mathbf{L} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \otimes \mathbf{I}_d. \tag{5}$$

Proof One can locate the new vertex of an equilateral triangle, inwardly or outwardly, constructed on one side of $\triangle_{\mathbf{x}}$ in the plane containing $\mathbf{x}$ using different geometric properties of equilateral triangles. We find it convenient to use the perpendicular bisector of the corresponding side of $\triangle_{\mathbf{x}}$, the line passing through its midpoint and being perpendicular to it, such that the new vertex is on this bisector and at a proper distance away from the side of $\triangle_{\mathbf{x}}$.

For instance, let $\mathbf{y} = [\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3]^{\mathrm{T}} = \mathrm{T}_{+}(\mathbf{x})$. Consider the side of $\triangle_{\mathbf{x}}$ joining x_1 and x_2, using the midpoint $\mathrm{m}_{12} := \frac{1}{2}(\mathrm{x}_1 + \mathrm{x}_2)$, to locate the new vertex, y_3, of inwardly erected triangle on this side as

$$\mathrm{y}_3 = \mathrm{m}_{12} + \frac{\sqrt{3}}{2}\mathbf{R}_{\mathbf{x}}(\mathrm{x}_2 - \mathrm{x}_1)\,, \tag{6}$$

where $\mathbf{R}_{\mathbf{x}}$ (see (1)) is a counter-clockwise rotation by $\frac{\pi}{2}$ in the plane where $\mathbf{x}$ is positively oriented. Note that the height of an equilateral triangle from any side is $\frac{\sqrt{3}}{2}$ times its side length. Hence, by symmetry, one can conclude (3).

Given a Torricelli configuration $\mathbf{y} = [\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3]^{\mathrm{T}} = \mathrm{T}_{\pm}(\mathbf{x})$, by definition, the vertices of the associated Napoleon triangle $\mathbf{z} = [\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3]^{\mathrm{T}} = \mathrm{N}_{\pm}(\mathbf{x})$ are given by

$$\mathrm{z}_1 = \frac{1}{3}(\mathrm{y}_1 + \mathrm{x}_2 + \mathrm{x}_3)\,,\ \mathrm{z}_2 = \frac{1}{3}(\mathrm{x}_1 + \mathrm{y}_2 + \mathrm{x}_3) \ \text{ and } \ \mathrm{z}_3 = \frac{1}{3}(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{y}_3)\,, \tag{7}$$

which is equal to (4), and so the result follows. $\square$

Note that the Torricelli and Napoleon transformations of $\mathbf{x}$ are unique if and only if $\mathbf{x} \in \mathbb{R}^{3d}$ is non-collinear. If, contrarily, $\mathbf{x}$ is collinear, then $\triangle_{\mathbf{x}}$ is both positively and negatively oriented, and for $d \geq 3$ there is more than one 2-dimensional subspace of $\mathbb{R}^d$ containing $\mathbf{x}$.

Remark 2.1 ([4]) For any $\mathbf{x} = [\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]^\mathrm{T} \in \mathbb{R}^{3d}$, the centroid of the Torricelli configuration $\mathbf{y} = [\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3]^\mathrm{T} = \mathrm{T}_\pm(\mathbf{x})$, the Napoleon configuration $\mathbf{z} = \mathrm{N}_\pm(\mathbf{x})$ and the original triple $\mathbf{x}$ all coincide, i.e.,

$$\mathrm{c}(\mathbf{x}) = \mathrm{c}(\mathbf{y}) = \mathrm{c}(\mathbf{z}) \,, \tag{8}$$

and the distances between the associated elements of $\mathbf{x}$ and $\mathbf{y}$ are all the same, i.e., for any $i \neq j \in \{1, 2, 3\}$

$$\left\|\mathrm{y}_i - \mathrm{x}_i\right\|_2 = \left\|\mathrm{y}_j - \mathrm{x}_j\right\|_2 . \tag{9}$$

An observation key to all further results is that Napoleon transformations of equilateral triangles are very simple.

Lemma 2.2 *The inner Napoleon transformation* N_+ *of any triple* $\mathbf{x} = [\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]^\mathrm{T} \in \mathbb{R}^{3d}$ *comprising the vertices of an equilateral triangle* $\triangle_\mathbf{x}$ *collapses it to the trivial triangle all of whose vertices are located at its centroid* $\mathrm{c}(\mathbf{x})$*,*

$$\mathrm{N}_+(\mathbf{x}) = \mathbf{1}_3 \otimes \mathrm{c}(\mathbf{x}) \,, \tag{10}$$

whereas the outer Napoleon transformation N_- *reflects the vertices of* $\triangle_\mathbf{x}$ *with respect to its centroid* $\mathrm{c}(\mathrm{x})$*,*

$$\mathrm{N}_-(\mathbf{x}) = 2 \cdot \mathbf{1}_3 \otimes \mathrm{c}(\mathbf{x}) - \mathbf{x}. \tag{11}$$

Here, $\mathbf{1}_3$ *is the* $\mathbb{R}^3$ *column vector of all ones, and* $\cdot$ *denotes the standard entrywise (or* Hadamard [12, Section 5.7]*) product.*

Proof Observe that the inwardly erected triangle on any side of an equilateral triangle is equal to the equilateral triangle itself, i.e., $\mathrm{T}_+(\mathbf{x}) = \mathbf{x}$, and so, by definition, one has (10). Alternatively, using (4), one can obtain

$$\mathrm{N}_+(\mathbf{x}) = \frac{1}{3}\big(\mathbf{K}\mathbf{x} + \mathrm{T}_+(\mathbf{x})\big) = \frac{1}{3}(\mathbf{K}\mathbf{x} + \mathbf{x}) = \mathbf{1}_3 \otimes \mathrm{c}(\mathbf{x}) \,, \tag{12}$$

where $\mathbf{K}$ is defined as in (5).

Now consider outwardly erected equilateral triangles on the sides of an equilateral triangle, and let $\mathbf{y} = [\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3]^\mathrm{T} = \mathrm{T}_-(\mathbf{x})$. Note that each erected triangle has a common side with the original triangle. Since $\triangle_\mathbf{x}$ is equilateral, observe that the midpoint of the unshared vertices of an erected triangle and the original triangle is equal to the midpoint of their common sides, i.e., $\frac{1}{2}(\mathrm{y}_1 + \mathrm{x}_1) = \frac{1}{2}(\mathrm{x}_2 + \mathrm{x}_3)$ and so on. Hence, we have $\mathrm{T}_-(\mathbf{x}) = \mathbf{K}\mathbf{x} - \mathbf{x}$. Thus, one can verify the result using (4) as

$$\mathrm{N}_-(\mathbf{x}) = \frac{1}{3}\big(\mathbf{K}\mathbf{x} + \mathrm{T}_-(\mathbf{x})\big) = \frac{1}{3}(\mathbf{K}\mathbf{x} + \mathbf{K}\mathbf{x} - \mathbf{x}) = 2 \cdot \mathbf{1}_3 \otimes \mathrm{c}(\mathbf{x}) - \mathbf{x}. \tag{13}$$

□

Since the Napoleon transformation of any triangle results in an equilateral triangle, motivated from Lemma 2.2, we now consider the iterations of the Napoleon transformation. For any $k \geq 0$, let $\mathrm{N}_{\pm}^{k} : \mathbb{R}^{3d} \rightarrow \mathbb{R}^{3d}$ denote the *k-th Napoleon transformation* defined to be

$$\mathrm{N}_{\pm}^{k+1} := \mathrm{N}_{\pm} \circ \mathrm{N}_{\pm}^{k}, \tag{14}$$

where we set $\mathrm{N}_{\pm}^{0} := id$, and $id : \mathbb{R}^{3d} \rightarrow \mathbb{R}^{3d}$ is the *identity map* on $\mathbb{R}^{3d}$.

It is evident from Lemma 2.2 that the following lemma holds.

Lemma 2.3 *For any $\mathbf{x} \in \mathbb{R}^{3d}$ and $k \geq 1$,*

$$\mathrm{N}_{+}^{k+1}(\mathbf{x}) = \mathbf{1}_3 \otimes \mathrm{c}(\mathbf{x}), \quad \textit{and} \quad \mathrm{N}_{-}^{k+2}(\mathbf{x}) = \mathrm{N}_{-}^{k}(\mathbf{x}). \tag{15}$$

As a result, the basis of iterations of the Napoleon transformations consists of $\mathrm{N}_{\pm}$ and $\mathrm{N}_{\pm}^{2}$, whose explicit forms, except N_{-}^{2}, are given above. Using (4) and (11), the closed form expression of the double outer Napolean transformation N_{-}^{2} can be obtained as

Lemma 2.4 *An arbitrary triple $\mathbf{x} = [\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]^{\mathrm{T}} \in \mathbb{R}^{3d}$ gives rise to the double outer Napoleon triangle, $\mathrm{N}_{-}^{2} : \mathbb{R}^{3d} \rightarrow \mathbb{R}^{3d}$, according to the formula*

$$\mathrm{N}_{-}^{2}(\mathbf{x}) = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathrm{T}_{+}(\mathbf{x}). \tag{16}$$

Proof By Napoleon's theorem, $\mathrm{N}_{-}(\mathbf{x})$ is an equilateral triangle. Using (4) and Lemma 2.2, one can obtain the result as follows:

$$\mathrm{N}_{-}^{2}(\mathbf{x}) = \mathrm{N}_{-}\big(\mathrm{N}_{-}(\mathbf{x})\big) = 2 \cdot \mathbf{1}_3 \otimes \mathrm{c}(x) - \mathrm{N}_{-}(\mathbf{x}) = 2 \cdot \mathbf{1}_3 \otimes \mathrm{c}(x) - \frac{1}{3}\big(\mathbf{K}\mathbf{x} + \mathrm{T}_{-}(\mathbf{x})\big), \tag{17}$$

$$= \frac{2}{3}(\mathbf{K}\mathbf{x} + \mathbf{x}) - \frac{1}{3}\big(\mathbf{K}\mathbf{x} + \mathrm{T}_{-}(\mathbf{x})\big) = \frac{2}{3}\mathbf{x} + \frac{1}{3}\big(\mathbf{K}\mathbf{x} - \mathrm{T}_{-}(\mathbf{x})\big) = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathrm{T}_{+}(\mathbf{x}), \tag{18}$$

where $\mathbf{K}$ is defined as in (5). □

Note that $\mathrm{N}_{-}^{2}(\mathbf{x})$ is a convex combination of $\mathbf{x}$ and $\mathrm{T}_{+}(\mathbf{x})$, see Figure 2.

3 Optimality of Napoleon Transformations

To best of our knowledge, the Napoleon transformation $\mathrm{N}_{\pm}$ is mostly recognized as being a function into the space of equilateral triangles. In addition to this inherited property, $\mathrm{N}_{\pm}^{2}$ has an optimality property that is not immediately obvious. Although the double inner Napoleon transformation N_{+}^{2} is not really that interesting to work with, it gives a hint about the optimality of N_{-}^{2}: for any given triangle N_{+}^{2} yields a trivial triangle, all of whose vertices are located at the centroid of the given triangle which, by definition, minimizes the sum of squared distances to the vertices of the original triangle. Surprisingly, one has a similar optimality property for N_{-}^{2}:

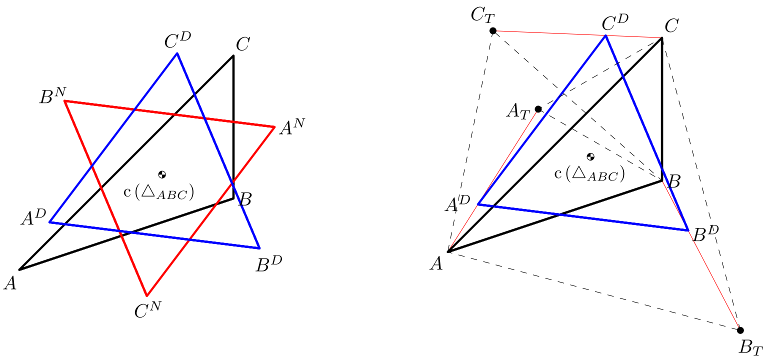

Fig. 2: (left) Outer, $\triangle_{A^N B^N C^N}$, and double outer, $\triangle_{A^D B^D C^D}$, Napoleon transformations of a triangle $\triangle_{ABC}$. (right) The double outer Napoleon triangle $\triangle_{A^D B^D C^D}$ is a convex combination of the original triangle $\triangle_{ABC}$ and the vertex set of its inner Torricelli configuration $\triangle_{A_T B_T C_T}$.

Theorem 3.1 *The double outer Napoleon transformation* $\mathrm{N}^2_-(\mathbf{x})$ *given in* (16) *yields the equilateral triangle most closely aligned with* $\triangle_{\mathbf{x}}$ *in the sense that it minimizes the total sum of squared distances between corresponding vertices. That is to say, for any* $\mathbf{x} = [\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]^{\mathrm{T}} \in \mathbb{R}^{3d}$, $\mathrm{N}^2_-(\mathbf{x})$ *is an optimal solution of the following problem:*

$$\begin{aligned} &\textit{minimize} \quad \sum_{i=1}^{3} \left\|\mathrm{x}_i - \mathrm{y}_i\right\|^2 \\ &\textit{subject to} \quad \left\|\mathrm{y}_1 - \mathrm{y}_2\right\|^2 = \left\|\mathrm{y}_1 - \mathrm{y}_3\right\|^2 = \left\|\mathrm{y}_2 - \mathrm{y}_3\right\|^2, \end{aligned} \tag{19}$$

where $\mathbf{y} = [\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3]^{\mathrm{T}} \in \mathbb{R}^{3d}$. *Furthermore, if* $\mathbf{x}$ *is non-collinear, then* (19) *has a unique solution.*

Proof Using the method of Lagrange multipliers [13], we first show that an optimal solution of (19) lies in the plane containing the triangle $\triangle_{\mathbf{x}}$. Then, to show the result, we solve (19) using a proper parametrization of equilateral triangles in $\mathbb{R}^2$.

The Lagrangian formulation of (19) minimizes

$$\begin{aligned} L(\mathbf{y}, \lambda_1, \lambda_2) = \sum_{i=1}^{3} \left\|\mathrm{x}_i - \mathrm{y}_i\right\|_2^2 + \lambda_1 \Big(\left\|\mathrm{y}_1 - \mathrm{y}_2\right\|_2^2 - \left\|\mathrm{y}_1 - \mathrm{y}_3\right\|_2^2 \Big) \\ + \lambda_2 \Big(\left\|\mathrm{y}_1 - \mathrm{y}_2\right\|_2^2 - \left\|\mathrm{y}_2 - \mathrm{y}_3\right\|_2^2 \Big), \end{aligned} \tag{20}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multipliers. A necessary condition for optimality in (19) is that the gradient $\nabla_{\mathbf{y}} L(\mathbf{y}, \lambda_1, \lambda_2)$ of the Lagrangian with respect to $\mathbf{y}$ at any locally optimal solution is zero,

$$\nabla_{\mathbf{y}} L(\mathbf{y}, \lambda_1, \lambda_2) = 2 \begin{bmatrix} (\mathrm{y}_1 - \mathrm{x}_1) + \lambda_1(\mathrm{y}_3 - \mathrm{y}_2) + \lambda_2(\mathrm{y}_1 - \mathrm{y}_2) \\ (\mathrm{y}_2 - \mathrm{x}_2) + \lambda_1(\mathrm{y}_2 - \mathrm{y}_1) + \lambda_2(\mathrm{y}_3 - \mathrm{y}_1) \\ (\mathrm{y}_3 - \mathrm{x}_3) - \lambda_1(\mathrm{y}_3 - \mathrm{y}_1) - \lambda_2(\mathrm{y}_3 - \mathrm{y}_2) \end{bmatrix} = 0, \tag{21}$$

from which one can conclude that an optimal solution of (19) lies in the plane containing $\triangle_{\mathbf{x}}$. Accordingly, without any loss of generality, suppose that $\triangle_{\mathbf{x}}$ is a positively oriented triangle in $\mathbb{R}^2$, i.e., its vertices are in counter-clockwise order in $\mathbb{R}^2$.

In general, an equilateral triangle $\triangle_{\mathbf{y}}$ in $\mathbb{R}^2$ with vertices $\mathbf{y} = [\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3]^{\mathrm{T}} \in \mathbb{R}^6$ can be uniquely parametrized using two of its vertices, say y_1 and y_2, and a binary variable $k \in \{-1, +1\}$ specifying the orientation of $\triangle_{\mathbf{y}}$; for instance, $k = +1$ if $\triangle_{\mathbf{y}}$ is positively oriented, and so on. Consequently, the remaining vertex, y_3, can be located as

$$\mathrm{y}_3 = \frac{1}{2}(\mathrm{y}_1 + \mathrm{y}_2) + k\frac{\sqrt{3}}{2}\mathbf{R}_{\pi/2}(\mathrm{y}_2 - \mathrm{y}_1), \tag{22}$$

where $\mathbf{R}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the rotation matrix defining a rotation by $\pi/2$.

Hence, one can rewrite the optimization problem (19) in terms of new parameters as an unconstrained optimization problem: for $\mathrm{y}_1, \mathrm{y}_2 \in \mathbb{R}^2$ and $k \in \{-1, 1\}$,

$$\text{minimize} \quad \|\mathrm{x}_1 - \mathrm{y}_1\|_2^2 + \|\mathrm{x}_2 - \mathrm{y}_2\|_2^2 + \left\|\mathrm{x}_3 - \mathbf{M}\mathrm{y}_1 - \mathbf{M}^{\mathrm{T}}\mathrm{y}_2\right\|_2^2 \,, \tag{23}$$

where $\mathbf{M} := \frac{1}{2}\mathbf{I} - k\frac{\sqrt{3}}{2}\mathbf{R}_{\pi/2}$, and $\mathbf{I}$ is the 2×2 identity matrix. Note that $\mathbf{M} + \mathbf{M}^{\mathrm{T}} = \mathbf{I}$, $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{M}\mathbf{M}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{M}^2 = -\mathbf{M}^{\mathrm{T}}$.

For a fixed $k \in \{-1, 1\}$, (23) is a convex optimization problem of y_1 and y_2, because every norm on $\mathbb{R}^n$ is convex, and compositions of convex functions with affine transformations preserve convexity [14]. Hence, a global optimal solution of (23) occurs where the gradient of the objective function is zero at

$$\begin{bmatrix} \left(\mathbf{I} + \mathbf{M}^{\mathrm{T}}\mathbf{M}\right) & \left(\mathbf{M}^2\right)^{\mathrm{T}} \\ \mathbf{M}^2 & \left(\mathbf{I} + \mathbf{M}\mathbf{M}^{\mathrm{T}}\right) \end{bmatrix} \begin{bmatrix} \mathrm{y}_1 \\ \mathrm{y}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{x}_1 + \mathbf{M}^{\mathrm{T}}\mathrm{x}_3 \\ \mathrm{x}_2 + \mathbf{M}\mathrm{x}_3 \end{bmatrix}, \tag{24}$$

which simplifies to

$$\begin{bmatrix} 2\mathbf{I} & -\mathbf{M} \\ -\mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathrm{y}_1 \\ \mathrm{y}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{x}_1 + \mathbf{M}^{\mathrm{T}}\mathrm{x}_3 \\ \mathrm{x}_2 + \mathbf{M}\mathrm{x}_3 \end{bmatrix}. \tag{25}$$

Note that the objective function, $f(\mathbf{y})$, is strongly convex, because its Hessian, $\nabla^2 f(\mathbf{y})$, satisfies

$$\nabla^2 f(\mathbf{y}) = \begin{bmatrix} 2\mathbf{I} & -\mathbf{M} \\ -\mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \succeq \mathbf{I}, \tag{26}$$

which means that for a fixed $k \in \{-1, +1\}$ the optimal solution of (23) is unique.

Now observe that

$$\frac{1}{3}\begin{bmatrix} 2\mathbf{I} & \mathbf{M} \\ \mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} \begin{bmatrix} 2\mathbf{I} & -\mathbf{M} \\ -\mathbf{M}^{\mathrm{T}} & 2\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{27}$$

hence the solution of the linear equation (25) is

$$\begin{bmatrix}\mathrm{y}_1\\ \mathrm{y}_2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\mathbf{I} & \mathbf{M}\\ \mathbf{M}^{\mathrm{T}} & 2\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathrm{x}_1+\mathbf{M}^{\mathrm{T}}\mathrm{x}_3\\ \mathrm{x}_2+\mathbf{M}\mathrm{x}_3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\mathrm{x}_1+2\mathbf{M}^{\mathrm{T}}\mathrm{x}_3+\mathbf{M}\mathrm{x}_2+\mathbf{M}^2\mathrm{x}_3\\ \mathbf{M}^{\mathrm{T}}\mathrm{x}_1+\left(\mathbf{M}^2\right)^{\mathrm{T}}\mathrm{x}_3+2\mathrm{x}_2+2\mathbf{M}\mathrm{x}_3\end{bmatrix}, \tag{28}$$

$$= \frac{1}{3}\begin{bmatrix}2\mathrm{x}_1+\mathbf{M}^{\mathrm{T}}\mathrm{x}_3+\mathbf{M}\mathrm{x}_2\\ 2\mathrm{x}_2+\mathbf{M}^{\mathrm{T}}\mathrm{x}_1+\mathbf{M}\mathrm{x}_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\left(\mathrm{x}_1+\frac{\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_3}{3}\right)+k\frac{1}{2\sqrt{3}}\mathbf{R}_{\pi/2}(\mathrm{x}_3-\mathrm{x}_2)\\ \frac{1}{2}\left(\mathrm{x}_2+\frac{\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_3}{3}\right)+k\frac{1}{2\sqrt{3}}\mathbf{R}_{\pi/2}(\mathrm{x}_1-\mathrm{x}_3)\end{bmatrix}. \tag{29}$$

Here, substituting y_1 and y_2 back into (22) yields

$$\mathrm{y}_3 = \frac{1}{2}\left(\mathrm{x}_3 + \frac{\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_3}{3}\right) + k\frac{1}{2\sqrt{3}}\mathbf{R}_{\pi/2}(\mathrm{x}_2-\mathrm{x}_1)\,. \tag{30}$$

Thus, overall, we have

$$\mathbf{y} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\left(\frac{1}{2}\mathbf{K}\mathbf{x} + k\frac{\sqrt{3}}{2}\left(\mathbf{I}_3\otimes\mathbf{R}_{\pi/2}\right)\mathbf{L}\mathbf{x}\right) = \begin{cases}\frac{2}{3}\mathbf{x}+\frac{1}{3}\mathrm{T}_+(\mathbf{x}), & \text{if } k=+1,\\ \frac{2}{3}\mathbf{x}+\frac{1}{3}\mathrm{T}_-(\mathbf{x}), & \text{if } k=-1,\end{cases} \tag{31}$$

where $\mathbf{K}$ and $\mathbf{L}$ are defined as in (5). Recall that $\triangle_{\mathbf{x}}$ is assumed to be positively oriented, i.e., $\mathbf{R}_{\mathbf{x}} = \mathbf{R}_{\pi/2}$, and so it is convenient to have the results in terms of Torricelli transformations $\mathrm{T}_\pm$, see (3). As a result, the difference of $\mathbf{y}$ and $\mathbf{x}$ is simply given by

$$\mathbf{y}-\mathbf{x} = \begin{cases}\frac{1}{3}\left(\mathrm{T}_+(\mathbf{x})-\mathbf{x}\right), & \text{if } k=+1,\\ \frac{1}{3}\left(\mathrm{T}_-(\mathbf{x})-\mathbf{x}\right), & \text{if } k=-1.\end{cases} \tag{32}$$

Finally, one can easily verify that the optimum value of k is equal to $+1$, since the distance of $\mathbf{x}$ to the vertices of its inner Torricelli configuration $\mathrm{T}_+(\mathbf{x})$ is always less than or equal to its distance to the vertices of its outer Torricelli configuration $\mathrm{T}_-(\mathbf{x})$. Here, the equality only holds if $\mathbf{x}$ is collinear. Thus, an optimal solution of (19) coincides with the double outer Napoleon transformation, $\mathrm{N}_-^2(\mathbf{x})$ (16), and it is the unique solution of (19) if $\mathbf{x}$ is non-collinear. □

4 Conclusions

In this paper, we present an interesting, but not so obvious, optimality property of Napoleon transformations: an optimally aligned equilateral triangle with any given triangle that minimizes sum of squared distances between the corresponding vertices of triangles is the double outer Napoleon transformation of the original triangle. An open question is whether any extension of Napoleon's theorem to higher dimensional simplices [9,10] has a similar optimality property.

It is also useful to note that our particular interest in the optimality of Napoleon triangles comes from our research on coordinated robot navigation, where a group of robots require to interchange their (structural) adjacencies through a minimum cost configuration, determined by the double outer Napoleon transformation [15]. We are currently exploring the use of Napoleon transformations for optimal pattern formation control of multirobot systems for applications such as search and rescue, area exploration, surveillance and reconnaissance, and environment monitoring.

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