Hierarchically Clustered Navigation of Distinct Euclidean Particles

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***Abstract*— This paper introduces and solves the problem of cluster-hierarchy-invariant particle navigation in $\mathrm{Conf}\big(\mathbb{R}^d, J\big)$. Namely, we are given a desired goal configuration, $\mathbf{x}^* \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$ and τ, a specified cluster hierarchy that the goal supports. We build a hybrid closed loop controller guaranteed to bring any other configuration that supports τ to the desired goal, $\mathbf{x}^* \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$, through a transient motion whose each configuration along the way also supports that hierarchy.**

I. Introduction

Given an index set, $J = [n] := \{1, \ldots, n\} \subset \mathbb{N}$, a *configuration,* $\mathbf{x} = (\mathrm{x}_i)_{i \in J}$, is a labeled placement of $|J| = n$ distinct Euclidean particles, x_i. We find it convenient to identify the *configuration space* [1] with the set of distinct labelings, i.e., the injective mappings of J into $\mathbb{R}^d$,

$$\mathrm{Conf}\big(\mathbb{R}^d, J\big) := \left\{ \mathbf{x} \in \left(\mathbb{R}^d\right)^J \middle| \left\|\mathrm{x}_i - \mathrm{x}_j\right\| \neq 0, \forall i \neq j \in J \right\}.$$

A *clustering* is a partition of J, $\mathcal{I} \in \mathtt{Part}\,[J]$, induced by the relative loci of a configuration's particles, $(\mathrm{x}_i)_{i \in J} \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$, in a manner we shall make precise below in section II-B.3. A *hierarchy,* $\tau := \{\mathcal{I}_\ell\}_{0 \leq \ell \leq L} \subset \mathtt{Part}\,[J]$, is a list of partitions of J, ordered by *refinement* [1]: i.e. each $I \in \mathcal{I}_j$ has a "parent", $I \subseteq \tilde{I} \in \mathcal{I}_{j-1}$, and we declare by definition that always $\mathcal{I}_0 := \{J\}$. A *cluster hierarchy* is a hierarchy, τ on J, induced by a nested family of clusterings *supported* by the configuration — again in a manner made precise in section II-B.3.

This paper introduces and solves the problem of cluster-hierarchy-invariant particle navigation in $\mathrm{Conf}\big(\mathbb{R}^d, J\big)$. Namely, we are given a desired goal configuration, $\mathbf{x}^* \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$ and τ, a specified cluster hierarchy that the goal supports. We build a hybrid closed loop controller guaranteed to bring any other configuration that supports τ to the desired goal, $\mathbf{x}^* \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$, through a transient motion whose each configuration along the way also supports that hierarchy.

A. Motivation

It is of widespread interest in robotics and automation applications to generate vector field planners [2] capable of enforcing complex coordinated group tasks specified in language such as: "you even-numbered agents stick together near the goal box while the odd-numbered agents play small-group zone defense organized around their numerically closest prime-numbered agents." Such tasks are often amenable to specification using algebraic predicates involving relative distances, but working out a few simple examples will convince the reader that their complexity grows very quickly in the number of agents and branchiness of the desired hierarchy. Moreover, the generation of such predicates from high level specification is not simple (it essentially requires the kind of analysis we introduce in this paper), and, worse, it is particularly tricky to express the precise but non-rigid relationship implied by "stick together" in this manner. Of course, the virtue of such precise yet flexible task formulations is that they might likely be composable without (much) interference with further "orthogonal" task specifications. For example, perhaps the prime-numbered agents were given that position of local forward leadership because they enjoy the most acute sensory endowment and we might now like separately to instruct them to track any spherical white objects whizzing around their assigned zone.

Hierarchical motion planning represents an important open problem domain in robotics as reviewed with some care in [3]. In that work, a two layer hierarchy is maintained by recourse to a fine grained cellular decomposition of the configuration space permitting the systematic application of multiple gradient-style vector fields one "preparing" the next to form a "sequentially composed" [4] hybrid system using constructions in the manner introduced by [5]. Here, we address the problem of maintaining arbitrarily deep and complex hierarchies and seek to substitute analytical insight for computational effort, replacing the familiar fine-grained, numerically determined (contractible) cellular decomposition with a far coarser but topologically more intricate decomposition into "strata." We strongly believe (but do not explicitly address beyond some speculative concluding remarks) that the intrinsic properties of a hierarchy will make it possible to replace our present centralized (full state information about the entire group available instantaneously to each individual agent) algorithm with a provably correct distributed version.

There is a dual task domain wherein we expect the ability to "servo" around hierarchically specified configurations may open up new robotic applications. A huge literature of perceptual and learning algorithms is bound up in the systematic decomposition of erstwhile homogeneous feature vector clouds into clustered subgroups. Whereas fixed resolution clusters are known to preclude reasonably axiomatized foundations [6], cluster hierarchies similar (but, importantly, not identical) to the kind we introduce here have been shown to resolve such contradictions [7]. An early example suggesting the virtues of perceptual servoing [8] entails moving a robot around until the configuration of certain perceptually worthy feature vectors is "properly" arranged.

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[1] It will become clear in section II-B that ℓ is the "level" and $L \leq \ln n$ is the "depth" of the tree associated with this hierarchy.

We suspect there is far greater power and generality to be explored in servoing feature vector configurations into the sort of precise but flexible nested cluster relationships that form the central object of study here.

B. Contributions

The central achievement in this preliminary study is the construction of a centralized hybrid controller in section III that is shown to bring the entire "stratum" of configurations supporting a specified hierarchy to a goal configuration while preserving the hierarchy along the way. However, we are also very enthusiastic about the method by which these results obtain. Namely, the identification of a stratum's homotopy type (the generalized torus revealed in Theorem 1) suggests a change of coordinates into an appropriately "thickened" homotopy model wherein the business of manuevering the hierarchy (40b) can be nearly decoupled from the business of preserving it (40c).

II. Background, Notation and Preliminary Results

A. Hierarchies

1) Trees: A hierarchy, $\tau = \{\mathcal{I}_\ell\}_{0\leq\ell\leq L}$, uniquely determines (and henceforth will be conflated with) a tree — a loop-free graph whose vertices are identified with the constituent cells of the various partitions, $\mathcal{C}(\tau) :=\cup_{0\leq\ell\leq L}\mathcal{I}_\ell$ and whose edges represent the set inclusion relation. We will take the top cell, $\mathcal{I}_0 = \{J\}$, to be the root, and this induces an orientation on the graph, allowing us to speak of "out-" or "in-directed" edges as well as "levels" or "depth" down the tree.

We adopt the following notation

$$\begin{aligned}
Anc(I,\tau) &= \left\{V\in\mathcal{C}(\tau)\,\middle|\,I\subsetneq V\right\},\\
Pr(I,\tau)^{2} &\in \left\{V\in\mathcal{C}(\tau)\,\middle|\,V\supsetneq I,\ \nexists Y\in\mathcal{C}(\tau)\ \text{s.t.}\ I\subsetneq Y\subsetneq V\right\},\\
Ch(I,\tau) &= \left\{V\in\mathcal{C}(\tau)\,\middle|\,V\subset I,\ \nexists Y\in\mathcal{C}(\tau)\ \text{s.t.}\ V\subsetneq Y\subsetneq I\right\},\\
Des(I,\tau) &= \left\{V\in\mathcal{C}(\tau)\,\middle|\,V\subsetneq I\right\},
\end{aligned}$$

for the standard notions of, respectively, the set of ancestors, parents, children and descendants of a vertex in the tree. Because the children comprise a partition of each parent, we find it convenient to define a *local complementary* cluster, I^{LC}, of a cluster $I\in\mathcal{C}(\tau)$ as

$$I^{LC}\in\left\{V\in\mathcal{C}(\tau)\ \middle|\ Pr(I,\tau)=Pr(V,\tau),V\neq I\right\}, \quad (1)$$

not to be confused with the standard (global) complement, $I^C = J - I$, which is distinct (unless $Pr(I,\tau) = J$).

2) Nondegeneracy: A binary partition is called a *split*, and a tree is said to be *binary* or, equivalently, *nondegenerate* if the children of each parent node comprise a (local) split. All other trees are said to be *degenerate*. In this paper we will confine attention to nondegenerate hierarchies, a set of trees we denote as $\overline{\mathcal{T}}_J$, in a manner to be made precise in section II-B.3.

[2] For completeness the parent of the coarsest cluster is declared empty, $Pr(J,\tau)=\emptyset$.

A *terminal* vertex corresponds to a singleton cell (hence having out-degree zero), as opposed to an *interior* vertex corresponding to a non-singleton cell. For later use, we note the following fact, whose (omitted) proof can be established by induction.

Lemma 1 *Let $\tau\in\overline{\mathcal{T}}_J$ be a nondegenerate hierarchy over $J\in\mathbb{N}$. Then τ has $|J|-1$ interior vertices.*

Proof: See Appendix -A. ■

B. Clusters

1) Cluster Functions: Given $I\subseteq J$, denote by $\mathbf{x}|I$, the *partial configuration* obtained from the restricted labeling, $\mathbf{x}|I := (\mathrm{x}_i)_{i\in I}$, interpreted geometrically as a "cluster" of $|I|$ distinct points within the ambient space, $\mathbb{R}^d$. Using this notion, a configuration gives rise to a variety of useful "cluster functions, " including the centroid of any partial configuration, and, given two proper subsets, $A,B\subsetneq J$, the separation vector from the centroid of one $\mathbf{x}|A$ to that of the other $\mathbf{x}|B$ along with the midpoint of their centroids, defined, respectively, as

$$\begin{aligned}
\mathrm{c}(\mathbf{x}|I) &:= \frac{1}{|I|}\sum_{i\in I}\mathrm{x}_i, &(2)\\
\mathrm{s}(\mathbf{x};A;B) &:= \mathrm{c}(\mathbf{x}|B)-\mathrm{c}(\mathbf{x}|A), &(3)\\
\mathrm{m}(\mathbf{x};A;B) &:= \frac{\mathrm{c}(\mathbf{x}|A)+\mathrm{c}(\mathbf{x}|B)}{2}. &(4)
\end{aligned}$$

For the latter two of these cluster functions, in the context of a specified binary tree, τ, and specified split, $A = I, B = I^{LC}$, we will abuse notation via the shorthand, e.g., $\mathrm{m}_{I,\tau}(\mathbf{x}) := \mathrm{m}\left(\mathbf{x};I;I^{LC}\right)$, and $\mathrm{s}_{I,\tau}(\mathbf{x}) := \mathrm{s}\left(\mathbf{x};I;I^{LC}\right)$.

The scalar valued *separation magnitude* function [3]

$$\eta(\mathbf{x};i,A) := \langle x_i - \mathrm{m}(\mathbf{x};A;J\setminus A), \mathrm{s}(\mathbf{x};A;J\setminus A)\rangle \quad (5)$$

and its associated local split variant $\eta_{i,I,\tau}(\mathbf{x}) := \langle x_i - \mathrm{m}_{I,\tau}(\mathbf{x}), \mathrm{s}_{I,\tau}(\mathbf{x})\rangle$ will figure prominently throughout the sequel.

2) Central Voronoi Tesselations: We take the following definitions from [9] to which the reader is referred for a more careful presentation of these ideas. A *voronoi tessellation* (VT) is a decomposition of an open set $\Omega\subseteq\mathbb{R}^d$ via a metric, $d:\mathbb{R}^d\times\mathbb{R}^d\to\overline{\mathbb{R}_+}$ into *voronoi regions* (cells) $\{V_i\}_{i\in[k]}$,

$$V_i = \left\{\mathrm{x}\in\Omega \mid d(\mathrm{x},\mathrm{z_i}) < d(\mathrm{x},\mathrm{z_j}),\ \forall j\in[k]\right\},\ \text{for all } i\in[k], \quad (6)$$

around a finite set of generators (seeds), $\mathbf{z} = \{\mathrm{z}_i\}_{i\in[k]}$, contained in $\overline{\Omega}$ [9]. Using the voronoi tessellation of $\mathbb{R}^d$ with the standard Euclidean metric around a set of generators $\mathbf{z}$, a *k-partition* $\mathcal{I} = \{I_i\}_{i\in[k]}$ of the index set J of a configuration $\mathbf{x}\in\mathrm{Conf}\left(\mathbb{R}^d,J\right)$ is termed a *VT induced partition* of J if it satisfies

$$\|\mathrm{x}_l - \mathrm{z}_i\| \leq \|\mathrm{x}_l - \mathrm{z}_j\|,\ \forall\, l\in I_i,\ j\neq i\in[k]. \quad (7)$$

[3] Here, $\langle .,.\rangle$ denotes the inner product in a Euclidean space.

A *centroidal voronoi tessellation* (CVT) is a special type of a voronoi decomposition whose generators $\mathbf{z}$ coincide with the centroid for each block, $\mathrm{z}_i = \mathrm{z}_i^*$,

$$\mathrm{z}_i^* = \frac{1}{|I_i|} \sum_{l \in I_i} \mathrm{x}_l = \mathrm{c}\left(\mathbf{x}|I_i\right), \tag{8}$$

for the discrete case. A CVT can be computed via Lloyds algorithm [10] or k-means clustering [11] (a special case of Lloyds algorithm for a discrete set of data).

Definition 1 *A* CVT split *of a finite label set $J \subset \mathbb{N}$ of a set of points $\mathbf{x} = (\mathrm{x}_i)_{i \in J}$ in the Euclidean space $\mathbb{R}^d$ is a CVT induced binary partition, $\{I, J \setminus I\}$, of J, which by definition has the following property*

$$\eta\left(\mathbf{x}; i, A\right) \leq 0 \quad \forall\, i \in A,\ A \in \{I, J \setminus I\}, \tag{9}$$

A useful observation about a CVT split in a Eucliden space is that the voronoi tessellation generated by centroids of the partial configuration, $\mathbf{x}|I$,and its complementary configuration, $\mathbf{x}|J \setminus I$, decomposes the space into two half spaces by a hyperplane passing through the midpoint of centroids, $\mathrm{m}\left(\mathbf{x}; I; J \setminus I\right)$, and perpendicular to the vector between them, $\mathrm{s}\left(\mathbf{x}; I; J \setminus I\right)$.

3) CVT Hierarchies and Their Support: We adopt a divisive hierarchical clustering method based on centroidal voronoi tessellations known as "bisecting k-means" [12]. Briefly, this method splits each successive partial configuration by applying 2-means, and successively continues with each subsplit until reaching singletons. A hierarchy, $\mathcal{C}\left(\tau\right)$, generated in this manner thus determines a unique nondegenerate combinatorial tree $\tau \in \overline{\mathcal{T}}_J$, where $\overline{\mathcal{T}}_J$ is the set of all combinatorial trees with leaves injectively marked from J.

Definition 2 *A configuration $\mathbf{x} \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$ is said to* support *a non-degenerate hierarchy $\tau \in \overline{\mathcal{T}}_J$ if all of its splits in τ satisfy*

$$\eta_{i,I,\tau}\left(\mathbf{x}\right) \leq 0 \quad \forall\ \ i \in I, I \in \mathcal{C}\left(\tau\right) \setminus J. \tag{10}$$

The stratum *associated with τ is the set of all configurations that support it,*

$$\mathfrak{S}\left(\tau\right) = \left\{\mathbf{x} \in \mathrm{Conf}\big(\mathbb{R}^d, J\big) \;\middle|\; \mathbf{x} \textit{ supports } \tau\right\}. \tag{11}$$

C. Homotopy Type of a Nondegenerate Hierarchical Stratum

We now follow [13] in defining terminology and expresssions leading to the characterization of the homotopy type of the stratum, $\mathfrak{S}\left(\tau\right)$, associated with a nondegenerate hierarchy. The proofs of our formal statements all follow the same pattern as established in [13], and we omit them to save space here.

Definition 3 *A configuration $\mathbf{x} \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$ is* narrow *relative to the split, $\{I, J \setminus I\}$, if*

$$\max_{A \in \{I, J \setminus I\}} r_A(\mathbf{x}) \;<\; \frac{1}{2}\left\|\mathrm{s}\left(\mathbf{x}, I, J \setminus I\right)\right\|, \tag{12}$$

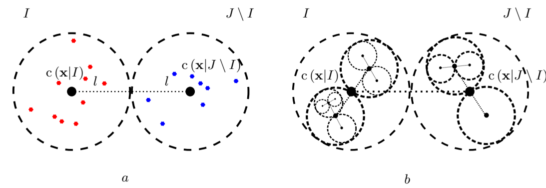

Fig. 1. Illustration of (a) narrow and (b) standard configurations.

where the radius of a cluster, $A \subset J$, is defined to be

$$r_A(\mathbf{x}) := \max_{i \in A} \|\mathrm{x}_{i,A}\|; \quad \mathrm{x}_{i,A} := \mathrm{x}_i - \mathrm{c}\left(\mathbf{x}|A\right). \tag{13}$$

Say that $\mathbf{x} \in \mathfrak{S}\left(\tau\right)$ is a standard *configuration relative to the nondegenerate hierarchy, $\tau \in \overline{\mathcal{T}}_J$, if it is narrow relative to each local split, $Ch\left(A, \tau\right)$ of every cluster, $A \in \mathcal{C}\left(\tau\right)$.*

Proposition 1 *If $\mathbf{x} \in \mathfrak{S}\left(\tau\right)$ is a standard configuration then for each cluster, $A \in \mathcal{C}\left(\tau\right)$, any rigid rotation of the partial configuration, $\mathbf{x}|A$, around its centroid, $\mathrm{c}\left(\mathbf{x}|A\right)$, preserves the supported hierarchy τ.*

Proposition 2 *Let $J \subset \mathbb{N}$ be a finite non-empty label set and suppose $\tau \in \overline{\mathcal{T}}_J$ be a non-degenerate tree. Then there exists a strong deformation retraction*

$$R_\tau : \mathfrak{S}\left(\tau\right) \times [0, 1] \rightarrow \mathfrak{S}\left(\tau\right) \tag{14}$$

of $\mathfrak{S}\left(\tau\right)$ onto the subset of standard configurations of $\mathfrak{S}\left(\tau\right)$.

These two observations now yield the key insight reported in [13].

Theorem 1 *The set of configurations $\mathbf{x} \in \mathrm{Conf}\big(\mathbb{R}^d, J\big)$ supporting a non-degenerate tree has the homotopy type of $\left(\mathbb{S}^{d-1}\right)^{|J|-1}$.*

Let us define open and closed strata of a nondegenerate hierarchy $\tau \in \overline{\mathcal{T}}_J$ as follows

$$\mathfrak{S}_o\left(\tau\right) = \bigcap_{I \in \mathcal{C}(\tau) \setminus \{J\}} \bigcap_{i \in I} \eta_{i,I,\tau}^{-1}(-\infty, 0), \tag{15}$$

$$\mathfrak{S}_{cl}\left(\tau\right) = \mathfrak{S}\left(\tau\right) = \bigcap_{I \in \mathcal{C}(\tau) \setminus \{J\}} \bigcap_{i \in I} \eta_{i,I,\tau}^{-1}(-\infty, 0], \tag{16}$$

where $\eta_{i,I,\tau}$ (5) is the separation magnitude.

Moreover, from the continuity of $\eta_{i,I,\tau}$, we have $\overline{\mathfrak{S}_o\left(\tau\right)} \subset \mathfrak{S}_{cl}\left(\tau\right)$, and a super set of the boundary of $\mathfrak{S}_o\left(\tau\right)$ is

$$\partial(\mathfrak{S}_o\left(\tau\right)) \subset \mathcal{H}^4 := \mathfrak{S}_{cl}\left(\tau\right) \setminus \mathfrak{S}_o\left(\tau\right). \tag{17}$$

[4] $\mathcal{H}$ contains all configurations with at least one agent on a separating hyperplane between clusters of τ, i.e. $\eta_{i,I,\tau} = 0$ for some cluster $I \in \mathcal{C}\left(\tau\right)$ and some $i \in I$.

III. PRINCIPAL RESULTS

Although first order vector field planners readily lift to second order [14] and a growing selection of under-actuated [8] mechanical plants, in this preliminary study we assume for ease of exposition a fully-actuated single-integrator point agent model for each particle. We further assume that some desired configuration, $\mathbf{x}^* \in \mathrm{Conf}\left(\mathbb{R}^d, J\right)$ has been selected, supporting some desired nondegenerate tree, τ. Our dynamical planner takes the form of a centralized hybrid controller, $\mathrm{u}: \mathfrak{S}_{cl}(\tau) \rightarrow \mathbb{R}^d$, for hierarchical navigation in $\mathfrak{S}_o(\tau)$ towards a desired configuration $\mathbf{x}^* \in \mathfrak{S}_o(\tau)$ as follows:

$$\dot{\mathrm{x}}_i = \mathrm{u}_i(\mathbf{x}) = \begin{cases} \mathrm{u}_i^A(\mathbf{x}) & \text{, if } \mathbf{x} \in \mathfrak{S}_{cl}(\tau) \setminus \mathcal{D}_E(\tau), \\ \mathrm{u}_i^E(\mathbf{x}) & \text{, if } \mathbf{x} \in \mathcal{D}_E(\tau). \end{cases} \quad (18)$$

Here $\mathbf{u}^A := \left(\mathrm{u}_i^A\right)_{i \in J} : \mathfrak{S}_{cl}(\tau) \rightarrow \mathbb{R}^d$ is an alignment field that moves its labeled particle so as to asymptotically align the separating hyperplane normals of all nonsingleton clusters in its ancestor set, $Anc(\{i\}, \tau)$, while preserving the hierarchy. In contrast, $\mathbf{u}^E := \left(\mathrm{u}_i^E\right)_{i \in J} : \mathcal{D}_E(\tau) \rightarrow \mathbb{R}^d$ is an expansion field guaranteed to drive asymptotically each agent directly towards its desired location once the separating hyperplanes are "sufficiently aligned." The criterion for "sufficient alignment" is specified by its prescribed domain as (see Figure III)

$$\begin{aligned} \mathcal{D}_E(\tau) = \Big\{ \mathbf{x} \in \mathfrak{S}_{cl}(\tau) \,\Big|\, & \left(\mathrm{x}_i - \mathrm{m}_{I,\tau}(\mathbf{x})\right)^{\mathrm{T}} \mathrm{s}_{I,\tau}(\mathbf{x}^*) \\ & + \left(\mathrm{x}_i^* - \mathrm{m}_{I,\tau}(\mathbf{x}^*)\right)^{\mathrm{T}} \mathrm{s}_{I,\tau}(\mathbf{x}) \leq 0, \\ & \forall i \in I, I \in \mathcal{C}(\tau) \setminus \{J\} \Big\}. \end{aligned} \quad (19)$$

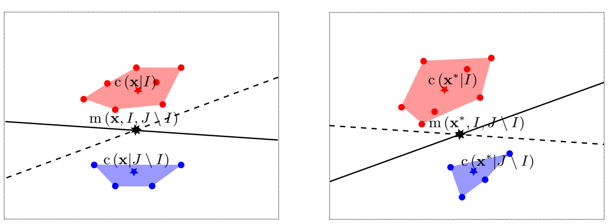

Fig. 2. An illustration of a configuration in the basin of attraction $\mathcal{D}_E(\tau)$ of the expansion field u^E. Both of the current (left) and desired (right) arrangements need to be linearly separable by each other's separating hyperplane, and this needs to be satisfied at each level of the hierarchy.

A. Alignment Field

1) Construction of the Controller: The alignment field, $\mathbf{u}^A : \mathfrak{S}_{cl}(\tau) \rightarrow \mathbb{R}^d$, is a (linear) combination of translation, rotation and scaling fields, $\{Tr, R, S\}$, associated with all clusters in $Anc(\{i\}, \tau)$ — that is, all the nonsingleton partial configurations of τ containing particle i,

$$\mathrm{u_i}^A(\mathbf{x}) = \mathrm{u}^{Tr}(\mathbf{x}) + \sum_{I \in Anc(\{i\},\tau)} \sum_{m \in \{R,S\}} \mathrm{u}_{i,I}^m(\mathbf{x}). \quad (20)$$

The term $\mathrm{u}_i^{Tr} : \mathfrak{S}_{cl}(\tau) \rightarrow \mathbb{R}^d$ is a common translation field asymptotically moving the centroid of the overall configuration to that of the desired configuration,

$$\mathrm{u}_i^{Tr}(\mathbf{x}) = -2\left(\mathrm{c}(\mathbf{x}|J) - \mathrm{c}(\mathbf{x}^*|J)\right). \quad (21)$$

Each term, $\mathrm{u}_{i,I}^R : \mathfrak{S}_{cl}(\tau) \rightarrow \mathbb{R}^d$ is a rotation field associated with ancestor I, designed to rigidly re-orient the entire partial configuration $\mathbf{x}|I$ into correspondence with its "subgoal" $\mathbf{x}^*|I$ by asymptotically aligning its separating hyperplane normal with the desired one,

$$\mathrm{u}_{i,I}^R(\mathbf{x}) = \mathbf{R}_I \mathrm{x}_{i,I}. \quad (22)$$

Here, $\mathrm{x}_{i,I}$ (13) is the centroidal displacement of x_i in cluster $I \in \mathcal{C}(\tau)$, and $\mathbf{R}_I$ is a (weighted planar) rotation matrix in the plane defined by present and desired normals n_I and n_I^*,

$$\mathbf{R}_I := [\mathrm{n}_I \; \mathrm{t}_I] \begin{bmatrix} 0 & -\omega_I \\ \omega_I & 0 \end{bmatrix} [\mathrm{n}_I \; \mathrm{t}_I]^{\mathrm{T}}. \quad (23)$$

These "normals" (the direction vectors of centroid difference), n_I and n_I^*, for the subsplits $\{I_1, I_2\} = Ch(I, \tau)$ for the current and desired configurations, respectively, are

$$\mathrm{n}_I := \mathrm{v} \circ \mathrm{s}(\mathbf{x}; I_1; I_2) \text{ and } \mathrm{n}_I^* := \mathrm{v} \circ \mathrm{s}(\mathbf{x}^*; I_1; I_2), \quad (24)$$

where the projection of $\mathbb{R}^d \setminus \{0\}$ onto $\mathbb{S}^{d-1}$ is defined as

$$\mathrm{v}(\mathrm{x}) := \frac{\mathrm{x}}{\|\mathrm{x}\|}. \quad (25)$$

To complete the account of (22), the planar rotation is specified by

$$\begin{aligned} \omega_I &:= 1 - \mathrm{n}_I^{*\mathrm{T}} \mathrm{n}_I, & (26) \\ \mathrm{t}_I &:= \mathbf{P}(\mathrm{n}_I)\, \mathrm{n}_I^* = \mathrm{n}_I^* - \mathrm{n}_I {\mathrm{n}_I}^{\mathrm{T}} \mathrm{n}_I^*, & (27) \end{aligned}$$

where $\mathbf{P}(\mathrm{n}_I)$, the projection onto the tangent space of $\mathbb{S}^{d-1}$ at point $\mathrm{n_I} \in \mathbb{S}^{d-1}$, $T_{\mathrm{n}_I}\mathbb{S}^{d-1}$, is

$$\mathbf{P}(\mathrm{n}_I) := \mathbf{I}_d - \mathrm{n}_I {\mathrm{n}_I}^{\mathrm{T}}, \quad (28)$$

and $\mathbf{I}_d$ is the $d \times d$ identity matrix. All in all, the rotation rate and the direction of rotation are $\omega_I \|\mathrm{t}_I\|$ and $\mathrm{v} \circ \mathrm{t}_I$ (if $\mathrm{t}_I \neq 0$, otherwise rotation rate is also zero), respectively.

Finally, $\mathrm{u}_{i,I}^S(\mathbf{x}) : \mathfrak{S}_{cl}(\tau) \rightarrow \mathbb{R}^d$ is a scaling field designed to guarantee the positive invariance of $\mathfrak{S}_o(\tau)$ by suitably "narrowing" any cluster as required along the way toward the subspace of standard configurations,

$$\mathrm{u}_{i,I}^S(\mathbf{x}) = \xi_I^R(\omega_I \|\mathrm{t}_I\|)\, \mathrm{x}_{i,I}. \quad (29)$$

Here $\xi_I^R : [0, \infty) \rightarrow (-\infty, 0]$ gives a proper scaling rate using geometry of the clusters (see Figure 3) to guarantee the preservation of hierarchy while rotating with a rate of ω,

$$\xi_I^R(\omega) := \begin{cases} -\dfrac{\sqrt{r_I(\mathbf{x})^2 - \gamma^2 r_{\mathcal{W}_I}(\mathbf{x})^2}}{\gamma r_{\mathcal{W}_I}(\mathbf{x})}\omega, & \text{if } r_I(\mathbf{x}) \geq \gamma r_{\mathcal{W}_I}(\mathbf{x}), \\ 0 & \text{, otherwise,} \end{cases} \quad (30)$$

where $\gamma \in (0,1)$ is a constant specifying how well the clusters of a standard configuration (see Definition 3) are separated, and $r_I : \mathfrak{S}_{cl}(\tau) \rightarrow \overline{\mathbb{R}_+}$ (13) is the radius of partial configuration $\mathbf{x}|I$, and $r_{\mathcal{W}_I} : \mathfrak{S}_{cl}(\tau) \rightarrow \overline{\mathbb{R}_+}$ is the radius of the largest closed ball centered at $\mathrm{c}(\mathbf{x}|I)$ and satisfying hierarchical constraints,

$$\begin{aligned} r_{\mathcal{W}_I}(\mathbf{x}) &= \min_{A \in (Anc(I,\tau) \setminus \{J\}) \cup \{I\}} -\left(\mathrm{c}(\mathbf{x}|I) - \mathrm{m}_{A,\tau}(\mathbf{x})\right)^{\mathrm{T}} \left(\mathrm{v} \circ \mathrm{s}_{A,\tau}(\mathbf{x})\right), \\ &= \min_{A \in (Anc(I,\tau) \setminus \{J\}) \cup \{I\}} \left| \left(\mathrm{c}(\mathbf{x}|I) - \mathrm{m}_{A,\tau}(\mathbf{x})\right)^{\mathrm{T}} \mathrm{n}_{Pr(A,\tau)} \right|. \end{aligned} \quad (31)$$

Note that r_{W_I} is closely related to the notion of a standard configuration as introduced in Definition 3.

To gain an intuitive appreciation for how this scaling field preserves the hierarchy, observe, if agent-i of cluster I is on a separating hyperplane of any ancestor cluster $A \in (Anc(I,\tau) \setminus \{J\}) \cup \{I\}$, i.e. $\eta_{i,A,\tau}(\mathbf{x}) = 0$, then we have

$$\left(\mathrm{u}^R_{i,I}(\mathbf{x}) + \mathrm{u}^S_{i,I}(\mathbf{x})\right)^{\mathrm{T}} \mathrm{s}_{A,\tau}(\mathbf{x}) < 0, \tag{32}$$

as can be visualized in Figure 3 by simply drawing a tangent line from a point on a separating hyperplane to the circle with radius γr_{W_I} located at $\mathrm{c}(\mathbf{x}|I)$, depicting the boundary of desired standard configurations.

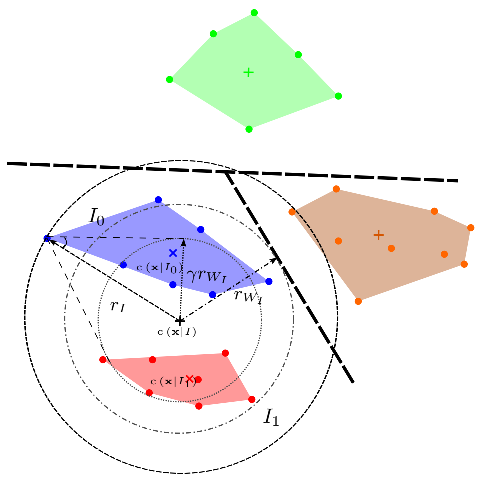

Fig. 3. An illustration of the geometric relation between the rotation and scaling fields in a planar environment. Their total effect is always tangent to the boundary of desired narrow configurations if a partial configuration is not narrow.

2) A Useful Change of Coordinates: The lexicographic order on the label set $J \subsetneq \mathbb{N}$ (the standard order relation over natural numbers) and depth-first traversal [5] of $\tau \in \mathcal{T}_J$ in inorder induce a hierarchical order over the cluster set $\mathcal{C}(\tau)$, recursively defined for clusters, $A, B \in \mathcal{C}(\tau)$, as:
(i)(base case) if $A = \emptyset$ or $A \cap B \neq \emptyset$,

$$A \leq B \text{ if } \begin{cases} A = B \text{ or} \\ A \subseteq Ch(B,\tau)_0 \text{ or} \\ B \subseteq Ch(A,\tau)_1, \end{cases} \tag{33}$$

(ii)(induction) otherwise

$$A \leq B \text{ if } A \leq Pr(B,\tau), \tag{34}$$

where the subscripts label elements of the split, $Ch(I,\tau) = \{Ch(I,\tau)_i\}_{i=0}^1$ with $\min(Ch(I,\tau)_0) < \min(Ch(I,\tau)_1)$. Using the induced hierarchical order over the singleton clusters, we have a hierarchical ordered label set $\tilde{J}$ with

$$\tilde{J}_i \leq \tilde{J}_j \text{ if } \{i\} \leq \{j\} \quad \forall i,j \in J. \tag{35}$$

[5]Depth-first (binary) tree traversal in inorder (symmetric) traverses the left subsplit, visits the parent and traverses the right subsplit [15].

Moreover, the cluster set $\mathcal{C}(\tau)$ satisfies

$$\mathcal{C}(\tau) = \mathcal{C}(\tau_0) \cup \{J\} \cup \mathcal{C}(\tau_1) \tag{36}$$

where $\mathcal{C}(\tau_0)$ and $\mathcal{C}(\tau_1)$ are the cluster set of subsplits $\{J_0, J_1\} = Ch(J,\tau)$, respectively,

$$\mathcal{C}(\tau_i) = \{I \cap J_i\}_{I \in \mathcal{C}(\tau)} \tag{37}$$

for $i = 1, 2$. Note that the hierarchical ordered $\mathcal{C}(\tau)$ is the same as the concatenation of $\mathcal{C}(\tau_0)$, $\{J\}$, $\mathcal{C}(\tau_1)$ in this order. This affords a canonical order on the $|J|-1$ interior clusters guaranteed by Lemma 1 to be well defined and fixed over the entirety of $\mathfrak{S}(\tau)$.

Define a linear transformation $f_\tau : (\mathbb{R}^d)^{|J|} \to (\mathbb{R}^d)^{|J|}$ that factors a configuration into its centroid and the $|J|-1$ centroid difference vectors associated with the nonsingleton partial configurations,

$$\begin{aligned} \mathbf{y} &= f_\tau(\mathbf{x}), \\ &= \left(\mathrm{c}(\mathbf{x}|J), \left(\mathrm{s}(\mathbf{x}; J_0; J_1)\right)_{\substack{A \in \mathcal{C}(\tau),\, |A|>1 \\ \{J_0,J_1\}=Ch(A,\tau)}}\right). \end{aligned} \tag{38}$$

Using the matrix representation of this transformation with respect to the canonical basis over the domain and its permutation (arising from the induced hierarchical order of interior clusters) over the codomain one proves by induction the following result.

Proposition 3 *The linear coordinate transformation $f_\tau : (\mathbb{R}^d)^{|J|} \to (\mathbb{R}^d)^{|J|}$ in (38) is bijective.*

Proof: See Appendix -B. ∎

Due to the convexity of k-means, the separation vectors of each nonsingleton cluster in $\mathfrak{S}_{cl}(\tau)$ are nonzero, hence the (nonlinear) restriction map $f_\tau|_{\mathfrak{S}_{cl}(\tau)}$ is a continuous bijection into its image in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})^{|J|-1}$. By recourse to the sphere map, v (25), we endow the punctured component of the image $f_\tau|_{\mathfrak{S}(\tau)}$ with spherical coordinates via the diffeomorphism

$$\begin{aligned} \mathbf{z} &= g_\tau(\mathbf{y}), \\ &= \left(\mathrm{y}_1, \left(\mathrm{v}(\mathrm{y}_A)\right)_{\substack{A\in\mathcal{C}(\tau) \\ |A|>1}}, \left(\|\mathrm{y}_A\|\right)_{\substack{A\in\mathcal{C}(\tau) \\ |A|>1}}\right). \end{aligned} \tag{39}$$

Because the restriction of $f_\tau|_{\mathcal{O}}$ to any open subspace $\mathcal{O} \subset \mathfrak{S}_{cl}(\tau)$ is a diffeomorphism, we can push forward the restricted alignment vector field $\mathrm{u}^A_i|_{\mathcal{O}}$ (20) via the diffeomorphism $g_\tau \circ f_\tau|_{\mathcal{O}}$ to reveal the simple form

$$\dot{\mathrm{z}}_1 = -2(\mathrm{z}_1 - \mathrm{z}^*_1), \tag{40a}$$

$$\dot{\mathrm{n}}_I = \sum_{A \in Anc(I,\tau) \cup \{I\}} \mathbf{R}_A \mathrm{n}_I, \tag{40b}$$

$$\dot{\rho}_I = \rho_I \sum_{A \in Anc(I,\tau) \cup \{I\}} \xi_A(\omega_A \|\mathrm{t}_A\|), \tag{40c}$$

permitting the following key observation.

Remark 1 *Since the (weighted planar) rotation matrix, $\mathbf{R}_I$ (23), is only a function of n_I and n_I^*, (40b) is an independent dynamical system describing the evolution of separating hyperplane normals, n_I.*

3) Invariance and Stability Properties: In this section we sketch the proof that: (i) a stratum, $\mathfrak{S}(\tau)$, is positive invariant under the flow of the alignment field, u_i^A; and (ii) it is included in the basin of attraction of a goal-aligned region of the standard configurations under that flow. [6]

Proposition 4 ([16]) *For a desired configuration $\mathbf{x}^* \in \mathfrak{S}_o(\tau)$ supporting a nondegenerate hierarchy $\tau \in \overline{\mathfrak{T}}_J$, the open pseudo-strata $\mathfrak{S}_o(\tau)$ is positive invariant under the alignment field, u^A (20).*

Proof: A sketch of the proof is as follows. $\mathcal{H}$ (17) is a super set of $\partial\mathfrak{S}_o(\tau)$ and it contains all configurations with $\eta_{i,I,\tau}(\mathbf{x}) = 0$ for some $I \in \mathcal{C}(\tau) \setminus \{J\}$ and some $i \in I$. Using (32), one can show that $\frac{\mathrm{d}}{\mathrm{d}t}\eta_{i,I,\tau}(\mathbf{x}) < 0$ on $\eta_{i,I,\tau}^{-1}[0]$ for any $I \in \mathcal{C}(\tau) \setminus \{J\}$ and any $i \in I$, which completes the proof. ■

Proposition 5 ([16]) *The target separating hyperplane normal $\mathbf{n}_\tau^* := (\mathrm{n}_I^*)_{I \in \mathcal{C}(\tau), |I|>1}$ associated with the goal configuration $\mathbf{x}^* \in \mathfrak{S}_o(\tau)$, via the map $g_\tau \circ f_\tau$ is an essentially globally asymptotically stable equilibrium state of the spherical dynamics (40b) whose basin excludes only the "antipodal fragments",*[7]

$$\left\{ \left(\mathbf{diag}\left[(-1)^{j_1}, (-1)^{j_2}, \ldots, (-1)^{j_{|J|-1}}\right] \otimes \mathbf{I}_d\right)\mathbf{n}_\tau^* \right| \left. (j_1, j_2, \ldots, j_{|J|-1}) \in \left(\{0,1\}^{|J|-1} - \{(0,0,\ldots,0)\}\right) \right\}$$

Proof: A sketch of the proof as follows. Consider a Lyapunov function $V : (\mathbb{S}^{d-1})^{|J|-1} \to \overline{\mathbb{R}_+}$

$$V(\mathbf{n}_\tau) = \sum_{\substack{I \in \mathcal{C}(\tau)\\ |I|>1}} \alpha_I U_I(\mathrm{n}_I), \tag{41}$$

where $(\alpha_I)_{\substack{I \in \mathcal{C}(\tau)\\ |I|>1}}$ are positive constants defined in (41) as a function of a height function, $U_I : \mathbb{S}^{d-1} \to [0,1]$, defined as

$$U_I(\mathrm{n}_I) = \frac{1}{2}(1 - {\mathrm{n}_I}^{\mathrm{T}}\mathrm{n}_I^*)^2, \tag{42}$$

Using the hierarchy τ, one can find proper constants α_I's such that $\dot{V}(\hat{\mathbf{n}}_\tau) = 0$ for all $\hat{\mathbf{n}}_\tau \in \mathfrak{N}(\tau)$ which also contains $\mathbf{n}_\tau^*$, and $\dot{V}(\mathbf{n}_\tau) < 0$ for all $\mathbf{n}_\tau \in (\mathbb{S}^{d-1})^{|J|-1} \setminus \mathfrak{N}(\tau)$,

$$\mathfrak{N}(\tau) = \left\{ \mathbf{n}_\tau \in (\mathbb{S}^{d-1})^{|J|-1} \middle| \mathrm{n}_I = \pm\mathrm{n}_I^*, \forall I \in \mathcal{C}(\tau), |I|>1 \right\}. \tag{43}$$

A simple computation shows that each point in the cardinality $2^{|J|-1}$ totally isolated set $\dot{V}^{-1}[0] = \mathfrak{N}(\tau)$ is repelling except for the atttractor, $\mathbf{n}_\tau^*$. It follows from LaSalle's Invariance Principle [17] that the basin of this attractor is the complement of $\mathfrak{N}(\tau) - \{\mathbf{n}_\tau^*\}$ as claimed. ■

Now consider the pre-image of the target separating hyperplane normal,

$$\mathcal{G}_A(\tau) := \left((g_\tau \circ f_\tau)^{-1}\{\mathrm{c}(\mathbf{x}^*|J)\} \times \{\mathbf{n}_\tau^*\} \times {\mathbb{R}_+}^{|J|-1}\right) \cap \mathfrak{S}_{cl}(\tau) \tag{44}$$

$$= \Big\{ \mathbf{x} \in \mathfrak{S}_{cl}(\tau) \Big| \mathrm{c}(\mathbf{x}|J) = \mathrm{c}(\mathbf{x}^*|J), \; \mathrm{n}_I = \mathrm{n}_I^* \; \forall I \in \mathcal{C}(\tau), |I| > 1 \Big\},$$

and recall that $\mathcal{D}_E(\tau)$ (19) contains configurations whose separating hyperplanes are nearly aligned with the desired one, and so $\mathcal{G}_A(\tau) \subset \mathrm{int}(\mathcal{D}_E(\tau))$. Pulling back a Lyapunov function for the attractor $\mathbf{n}_\tau^*$ of the spherical dynamics(40b) through the change of coordinates $g_\tau \circ f_\tau$ now yields a Lasalle function for $\mathcal{G}_A(\tau) \subset \mathrm{Conf}(\mathbb{R}^d, J)$, resulting in the desired conclusion.

Corollary 1 *The system in (20) converges from all configurations that support τ to a configuration in $\mathcal{G}_A(\tau) \subset \mathrm{int}(\mathcal{D}_E(\tau))$. Therefore, the alignment field, u^A (20), prepares the expansion field, u^E (45), in finite time.*

B. Expansion Field

1) General Form: The expansion field, $\mathrm{u}^E : \mathcal{D}_E(\tau) \to \mathbb{R}^d$, is simply designed as a negated gradient field of the sum of squared distances, $V(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^*\|^2$,

$$\mathrm{u}_i^E(\mathbf{x}) = -\nabla_{\mathrm{x}_i} V(\mathbf{x}) = -2(\mathrm{x}_i - \mathrm{x}_i^*), \tag{45}$$

such that it moves each agent directly towards to desired location and, looking ahead toward the proof of Proposition 6, we find it write out explicitly the exact trajectory followed by solutions of (45) as

$$\mathbf{x}^t = \mathbf{x}^* + e^{-2t}(\mathbf{x}^0 - \mathbf{x}^*) = e^{-2t}\mathbf{x}^0 + (1 - e^{-2t})\mathbf{x}^*, \tag{46}$$

where $\mathbf{x}^0 \in \mathcal{D}_E(\tau)$ denotes an initial configuration.

2) Invariance and Stability Properties:

Proposition 6 *For a desired configuration $\mathbf{x}^* \in \mathfrak{S}_o(\tau)$ supporting a nondegenerate hierarchy $\tau \in \overline{\mathfrak{T}}_J$, the basin of attraction, $\mathcal{D}_E(\tau)$ (19), of the expansion field, u^E (45), is positive invariant.*

Proof: One can simply verify that the domain condition holds along the trajectory (46) of expansion field u^E. In other words, for a CVT split $\{I, J \setminus I\}$, we have

$$\left(\mathrm{x}_i^t - \mathrm{m}(\mathbf{x}^t; A; J \setminus A)\right)^{\mathrm{T}} \mathrm{s}(\mathbf{x}^*; A; J \setminus A) + \left(\mathrm{x}_i^* - \mathrm{m}(\mathbf{x}^*; A; J \setminus A)\right)^{\mathrm{T}}, \mathrm{s}(\mathbf{x}^t; A; J \setminus A) \leq 0. \tag{47}$$

for all $i \in A$ and $A \in \{I, J \setminus I\}$ and $t \geq 0$. ■

[6] Unfortunately, $\mathrm{u}_i^A|_{\mathcal{O}}$ will generally not be complete on $\mathcal{O}$ so that we cannot work directly with the simple conjugate dynamics revealed just above.

[7] Here $\otimes$ is the standard Kronecker product, $\mathbf{I}_d$ is the $d \times d$ identity matrix, and $\mathbf{diag}[v]$ is the diagonal matrix whose diagonal is given by the vector whose entries are specified by the string v.

Proposition 7 *Any $\mathbf{x}^* \in \mathfrak{S}_o(\tau)$ is an asymptotically stable equilibrium point of the expansion field, u^E (45), whose basin of attraction includes $\mathcal{D}_E(\tau)$.*

Proof: It is obvious from the trajectory (46) of expansion field, u^E. ■

Corollary 2 *The hierarchy, τ, of a group of agents along the trajectory (46) of the expansion field (45) is preserved while reaching a desired configuration $\mathbf{x}^* \in \mathfrak{S}_o(\tau)$ from any initial configuration $\mathbf{x}^0 \in \mathfrak{S}_{cl}(\tau)$.*

Proof: One can easily check the CVT splits (Definition 1) along the trajectory of (46) of expansion field, u^E, and verify $\eta_{i,I,\tau}(\mathbf{x}^t) \leq 0$ for all $I \in \mathcal{C}(\tau) \setminus \{J\}$ and $i \in I$. ■

IV. CONCLUSION

In this paper, we introduce a novel application of clustering to the problem of precise yet flexible group coordination. We study a CVT based clustering scheme and reveal its homotopy model, affording a complete understanding of our navigation problem's topological complexity [18]. Based on this careful characterization of the underlying space, we introduce a centralized hybrid controller with provable global invariance and stability properties, which generalizes to an arbitrary number of particles and ambient space dimension.

These methods suggest a promising, broad domain of new hierarchical formation (motion) planning and hierarchical perceptual servoing problems.

In the near term, work in progress extends these ideas to navigation in the space of hierarchies to enable controllable switching between different hierarchies, i.e. hierarchical transitions to enable global coverage of $\mathrm{Conf}\left(\mathbb{R}^d, J\right)$. At the same time, a crucial requirement of practicable multi-agent systems is a distributed implementation of any coordination algorithm that drives a different slant of work in progress. Finally, we are exploring a number of application settings for hierarchical formation specification and control including problems of perception, perceptual servoing, anomaly detection and automated exploration and various problems of multi-agent coordination.

APPENDIX
PROOFS

A. Proof of Lemma 1

Proof:

The claim is trivially true for $n = 2$ (the root is the only interior vertex). For $n = 3$ there is only one binary tree: the root and its non-singleton child are the two interior vertices.

We proceed to the inductive step by considering an arbitrary binary tree, τ, now with more than 2 leaves and erase the root and its two outgoing edges to produce either: (i) a pair of binary trees, τ_0, τ_1 with n_0 and n_1 leaves each; or (ii) a binary tree, τ_0, and a one-vertex "tree."

In the first case, the inductive hypothesis affords the assumption that there are $n_j - 1$ interior vertice for tree τ_j, $j = 0, 1$, and their union, together with the original root, comprises the interior vertex set of τ which thus numbers $1+(n_0-1)+(n_1-1) = (1+n_0+n_1)-2 = (1+n)-2 = n-1$. In the second case, all interior vertices of τ except the root lie in τ_0. Since τ_0 has $n-1$ leaves, the inductive hypothesis yields the assumption that τ_0 has $n-2$ interior vertices. Hence, adding in the root, τ has $n-1$ interior vertices. ■

B. Proof of Proposition 3

Definition 4 *Associated with every n-leaf binary tree, τ, is a* hierarchy matrix *that we define inductively as follows. For $n = 1$, the one-vertex tree, τ, with no interior (root) vertex — define*

$$\mathbf{F}_\tau := [1] \,.$$

For $n \geq 2$ the hierarchy matrix is built using those associated with the subtrees, τ_0, τ_1 (possessing, respectively, n_0 and n_1 leaves), that arise when the root and its outgoing edges are removed from τ as follows: [8]

$$\mathbf{F}_\tau := \mathbf{diag}\left[n_0/n, \mathbf{1}_{n-1}\right]\mathbf{E}_\tau,$$

$$\mathbf{E}_\tau := \begin{bmatrix} \mathbf{F}_{\tau_0} & \mathbf{E}_{01} \\ -\mathbf{E}_{10} & \mathbf{F}_{\tau_1} \end{bmatrix}, \quad \mathbf{E}_{ij} := \frac{1}{n_0}\mathbf{e}_{n_i}\mathbf{1}_{n_j}^T.$$

Lemma 2 *The hierarchy matrix associated according to Definition 4 with an n-leaf tree, τ, has the two properties*

$$|\mathbf{F}_\tau| = 1; \quad \mathbf{1}_n^T \mathbf{F}_\tau{}^{-1}\mathbf{e}_n = n.$$

for all $n \in \mathbb{N}$.

Proof: The claim follows immediately for $n = 1$ where $\mathbf{F}_\tau = 1$. If τ has $n \geq 2$ leaves with first children τ_0, τ_1 then its associated hierarchy matrix has block determinant [19]

$$\begin{aligned}
|\mathbf{F}_\tau| &= \frac{n_0}{n}|\mathbf{F}_{\tau_0} + \mathbf{E}_{01}\mathbf{F}_{\tau_1}^{-1}\mathbf{E}_{10}| \cdot |\mathbf{F}_{\tau_1}| \\
&= \frac{n_0}{n}|\mathbf{F}_{\tau_0} + \frac{1}{n_0^2}\mathbf{e}_{n_0}\mathbf{1}_{n_1}^T\mathbf{F}_{\tau_1}^{-1}\mathbf{e}_{n_1}\mathbf{1}_{n_0}^T| \cdot |\mathbf{F}_{\tau_1}| \\
&= \frac{n_0}{n}|\mathbf{F}_{\tau_0} + \frac{\mathbf{1}_{n_1}^T\mathbf{F}_{\tau_1}^{-1}\mathbf{e}_{n_1}}{n_0^2}\mathbf{e}_{n_0}\mathbf{1}_{n_0}^T| \cdot |\mathbf{F}_{\tau_1}| \\
&= \frac{n_0}{n}|\mathbf{F}_{\tau_0}| \cdot |1 + \frac{\mathbf{1}_{n_1}^T\mathbf{F}_{\tau_1}^{-1}\mathbf{e}_{n_1}}{n_0^2}\mathbf{1}_{n_0}^T\mathbf{F}_{\tau_0}^{-1}\mathbf{e}_{n_0}| \cdot |\mathbf{F}_{\tau_1}| \\
&= \frac{n_0}{n}|\mathbf{F}_{\tau_0}| \cdot |1 + \frac{n_1}{n_0}| \cdot |\mathbf{F}_{\tau_1}| \\
&= 1
\end{aligned}$$

as claimed.

Moreover, we have

$$\begin{aligned}
\mathbf{F}_\tau{}^{-1} &= \mathbf{E}_\tau{}^{-1}\mathbf{diag}\left[n/n_0, \mathbf{1}_{n-1}\right] \\
\mathbf{E}_\tau{}^{-1} &= \begin{bmatrix} Q^{-1} & -Q^{-1}\mathbf{E}_{01}\mathbf{F}_{\tau_1}{}^{-1} \\ \mathbf{F}_{\tau_1}{}^{-1}\mathbf{E}_{10}Q^{-1} & \mathbf{F}_{\tau_1}{}^{-1}(I - \mathbf{E}_{10}Q^{-1}\mathbf{E}_{01}\mathbf{F}_{\tau_1}{}^{-1}) \end{bmatrix}
\end{aligned}$$

[8] Here $\mathbf{1}_k$ is the $\mathbb{R}^k$ column vector of all ones (hence $\mathbf{1}_0$ is defined to be the empty array) and $\mathbf{e}_k$ is the first canonical unit vector of Euclidean k-space (i.e. the column array whose first entry is unity and whose remaining $k-1$ entries are zeros).

where the Schur complement of the block $\mathbf{F}_{\tau_1}$ of the matrix $\mathbf{E}_\tau$ is

$$
\begin{aligned}
Q &:= \mathbf{F}_{\tau_0} + \mathbf{E}_{01}\mathbf{F}_{\tau_1}{}^{-1}\mathbf{E}_{10} \\
&= \mathbf{F}_{\tau_0} + \frac{1}{n_0^2}\mathbf{e}_{n_0}\mathbf{1}_{n_1}^T\mathbf{F}_{\tau_1}{}^{-1}\mathbf{e}_{n_1}\mathbf{1}_{n_0}^T \\
&= \mathbf{F}_{\tau_0} + \frac{n_1}{n_0^2}\mathbf{e}_{n_0}\mathbf{1}_{n_0}^T
\end{aligned}
$$

hence the matrix inversion lemma yields [19]

$$
\begin{aligned}
Q^{-1} &= \mathbf{F}_{\tau_0}{}^{-1} - \frac{n_1}{n_0^2}\frac{\mathbf{F}_{\tau_0}{}^{-1}\mathbf{e}_{n_0}\mathbf{1}_{n_0}^T\mathbf{F}_{\tau_0}{}^{-1}}{1+\frac{n_1}{n_0^2}\mathbf{1}_{n_0}^T\mathbf{F}_{\tau_0}{}^{-1}\mathbf{e}_{n_0}} \\
&= \mathbf{F}_{\tau_0}{}^{-1} - \frac{n_1}{n_0 n}\mathbf{F}_{\tau_0}{}^{-1}\mathbf{e}_{n_0}\mathbf{1}_{n_0}^T\mathbf{F}_{\tau_0}{}^{-1}
\end{aligned}
$$

and we can compute

$$
\begin{aligned}
\mathbf{1}_n^T\mathbf{F}_\tau^{-1}\mathbf{e}_n &= \frac{n}{n_0}\mathbf{1}_n^T\mathbf{E}_\tau^{-1}\mathbf{e}_n = \frac{n}{n_0}\left[\mathbf{1}_{n_0}^T, \mathbf{1}_{n_1}^T\right]\mathbf{E}_\tau^{-1}\begin{bmatrix}\mathbf{e}_{n_0}\\ \mathbf{0}_{n_1}\end{bmatrix} \\
&= \frac{n}{n_0}\left(\mathbf{1}_{n_0}^T Q^{-1}\mathbf{e}_{n_0} + \frac{1}{n_0}\mathbf{1}_{n_1}^T\mathbf{F}_{\tau_1}{}^{-1}\mathbf{e}_{n_1}\mathbf{1}_{n_0}Q^{-1}\mathbf{e}_{n_0}\right) \\
&= \frac{n^2}{n_0^2}\mathbf{1}_{n_0}^T Q^{-1}\mathbf{e}_{n_0} \\
&= \frac{n^2}{n_0^2}\mathbf{1}_{n_0}^T\left(\mathbf{F}_{\tau_0}{}^{-1} - \frac{n_1}{n_0 n}\mathbf{F}_{\tau_0}{}^{-1}\mathbf{e}_{n_0}\mathbf{1}_{n_0}^T\mathbf{F}_{\tau_0}{}^{-1}\right)\mathbf{e}_{n_0} \\
&= \frac{n^2}{n_0^2}\left(n_0 - \frac{n_1 n_0}{n}\right) \\
&= n,
\end{aligned}
$$

which completes the proof. ■

Corollary 3 *The map f_τ (38) has the nonsingular matrix representation $\mathbf{F}_\tau\mathbf{P}\otimes\mathbf{I}_d$, where $\mathbf{P}$ is a permutation matrix defined by the hierarchical ordered label set $\tilde{J}$ (35),*

$$
\mathbf{P}_{ij} = \begin{cases} 1 & , \text{if } \tilde{J}_i = j \\ 0 & , \text{otherwise.} \end{cases} \tag{48}
$$

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