TABLE III
Numerical Results

<table>
<thead>
<tr>
<th>Containership at 32 Knots</th>
<th>Solutions of the NCP</th>
<th>Eigenvalues of the Closed Loop System (A²)</th>
<th>Eigenvalues of P- Matrix:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A² =</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1768 x 10⁻¹</td>
<td>-0.1282 x 10²</td>
<td>0.1775 x 10⁻¹</td>
</tr>
<tr>
<td></td>
<td>-0.1218 x 10²</td>
<td>0.1266 x 10⁴</td>
<td>0.2988 x 10⁴</td>
</tr>
<tr>
<td></td>
<td>0.1775 x 10⁻¹</td>
<td>0.2286 x 10²</td>
<td>0.4257 x 10⁴</td>
</tr>
<tr>
<td></td>
<td>-0.1471 x 10⁻¹</td>
<td>-0.2828 x 10²</td>
<td>-0.5973 x 10⁴</td>
</tr>
<tr>
<td></td>
<td>-0.5973 x 10⁴</td>
<td>-0.8872 x 10⁴</td>
<td>-0.449 x 10⁴</td>
</tr>
<tr>
<td>State Feedback Gains (Stabilizing Control):</td>
<td>-0.5971 x 10⁻¹</td>
<td>-0.129 x 10²</td>
<td>0.100 x 10⁵</td>
</tr>
<tr>
<td></td>
<td>-0.1677 x 10²</td>
<td>0.1001 x 10⁰</td>
<td>-0.2370 x 10⁻²</td>
</tr>
<tr>
<td></td>
<td>-0.5973 x 10⁻¹</td>
<td>0.6565 x 10⁵</td>
<td>0.8567 x 10¹</td>
</tr>
</tbody>
</table>

The quadratic criterion of (7) is physically well motivated, with the weighting coefficient λ being completely defined a priori from the dynamics of the problem. It is possible that many other optimization problems can be successfully posed in this framework and solved in a more truly optimal manner rather than by the classical LR (Q ≥ 0) formulation where the quadratic weighting coefficients (generally) are iteratively specified by trial-and-error by the system designer on the basis of experimental studies.

REFERENCES

Stabilizability of Second-Order Bilinear Systems
DANIEL E. KODITSCHEK AND KUMPATHI N. NARENDRA

Abstract — This note states necessary and sufficient conditions for the existence of a linear state feedback controller such that a second-order bilinear system has a globally asymptotically stable closed loop. A suitable controller is constructed for each system which satisfies the conditions.

I. INTRODUCTION

This note concerns the stabilizability of second-order bilinear systems

\[ \dot{x} = A\dot{x} + Bu + \left( D\dot{x} + b \right) \]

where \( A, D \in \mathbb{R}^{2x2} \) and \( x, b \in \mathbb{R}^2 \). While a great amount of literature devoted to the structural properties of such systems has developed over the past decade [1]-[3], it is fair to say that it is understood regarding the qualitative behavior and trajectories of (1). Recently, several authors have investigated the stabilizability of systems of the form

\[ \dot{x} = Ax + \sum_{i=1}^{n} a_i (D_i x + b_i) \]

in \( \mathbb{R}^n \) [4], [6], [7]. These papers derive sufficient conditions and construct controllers to stabilize systems which meet specific and quite restrictive requirements. In our opinion, a significant understanding of bilinear systems will not be possible until more systematic analysis has been accomplished, and this note represents a step in that direction. Specifically, we give necessary and sufficient conditions for the existence of a constant linear feedback controller to stabilize (1). Even given the limited scope of this problem, it is safe to say that the statement of necessary and sufficient conditions is deceptively simple, and is possible only because of recent results in the stability of quadratic systems developed by the authors [5]. These results depend heavily upon that work.

Problem Statement: Characterize the properties of the triple \( (A, b, D) \) such that for some \( c \in \mathbb{R}^2 \), for \( u \neq c^T x \), the resulting second-order closed-loop system

\[ \dot{x} = A x + c^T x D x \quad A \triangleq A + bc^T \]

is globally asymptotically stable (GAS).

This problem is completely resolved by Theorem 1, stated below. It is worth remarking that a scalar bilinear system can never be made GAS using constant linear state feedback [8]; hence, the apparently restrictive conditions of Theorem 1 should not seem surprising. In the sequel, we assume that \( b \neq 0 \) and \( |D| \neq 0 \), and we will adopt the notation and definitions used in [5]. Briefly, \( [x, y] \) denotes the determinant of the array

\[ \begin{bmatrix} x \\ y \end{bmatrix} \]
Theorem 1: The triple \((A, b, D)\) is stabilizable under constant linear state feedback if and only if either

1. \(D\) has complex conjugate eigenvalues and \(x, y\), \(Jx, z\) has the same sign as \(|AD^{-1}b, b|\). If \(|AD^{-1}b, b| = 0\) then the special conditions given in Proposition 2, Section III hold or,

2. \(D\) is singular and its nonzero eigenvector is a stable eigenvector of \(A\).

If \(D\) is singular with a unique real eigenspace, then the special conditions given in Proposition 4, Section IV hold.

We present some preliminary results in Section II, then discuss condition i) of Theorem 1 in Section III, condition ii) in Section IV, and finally provide a proof of Theorem 1 by way of summary in Section V. A construction for a stabilizing linear constant controller is provided in the proof of each case, and reviewed in the summary.

II. Preliminary Results

Evidently, system (3) is an autonomous quadratic differential equation. In order to characterize the stabilizability of the triple \((A, b, D)\) under constant linear state feedback, we must, therefore, know something about the stability of such systems.

Theorem 2 [5]: System (3) is GAS if and only if:

1. \(A\), has eigenvalues with a nonpositive real part;
2. \([JD], [D'Jb, J]\), are sign definite or semidefinite with the same sign;
3. one of the following two mutually exclusive conditions holds:
   a) \(D\) is focal and \(D'\), is either focal or x-critical where \(x \in \mathbb{R}^n\) if \(|A| \neq 0\);
   b) \(D\) is x-critical and singular, \(|A| \neq 0\), and \(A^{-1}D = \gamma D\) for some scalar \(\gamma\).

The two distinct cases listed under condition iii) form a natural framework for the presentation of stabilizability conditions. In Section III, we discuss the properties of the triple \((A, b, D)\) when \(D\) is focal, corresponding to condition iii-a), above. In Section IV, we consider the case where \(D\) is singular, corresponding to condition iii-b), above. It is an immediate consequence of Theorem 2 that we need consider no other cases.

Lemma 1: If \(D\) is not focal and not singular, then system (1) cannot be stabilized by constant linear state feedback.

Proof: If \(D\) is nodal and nonsingular, then \([JD]\), is indefinite, and (3) violates condition ii) of Theorem 2 for any \(c \in \mathbb{R}^2\). If \(D\) is x-critical and nonsingular, then (3) violates condition iii) of Theorem 2 for any \(c \in \mathbb{R}^2\).

As a further consequence of Theorem 2, we must choose a linear control law, \(u = x^T, x\), for system (1) such that \([JD], [D'Jb, J]\), is sign definite or semidefinite depending upon the sign of \([JD]\). Thus, we may naturally inquire when a vector \(c \in \mathbb{R}^2\) exists such that \(D'JA, = D'JA + D'Jbc\) has a definite symmetric part. This question is resolved by the following lemma and its corollaries.

Lemma 2: For any \(Q \in \mathbb{R}^{2 \times 2}\) and \(c \in \mathbb{R}^2\), there exists a \(c \in \mathbb{R}^2\) such that \(|Q + gc, c| \geq 0\) if and only if \(g^T, Qg \geq 0\) if and only if \(Q + gc, c| \geq 0\).

Proof: i) Necessity: If \(g^T, Qg > 0\), then \(Q + gc, c| > 0\) if and only if \(g^T, Qg > 0\) if and only if \(g^T, Qg > 0\).

ii) Sufficiency: If \(g^T, Qg > 0\), then \(Q + gc, c| > 0\), for some \(d \in \mathbb{R}^2\). Hence, if \(c = -d + y\gamma\) for \(\gamma \in \mathbb{R}^n\), then \(|Q + gc, c| = |Q + gc, c| = y \gamma > 0\).

Note that any other choice of \(c\) leads to an indefinite form for \(|Q + gc, c|\).

Let \(g^T, Qg > 0\). Note that

\[
[(Q + gc, c)] = [Q, c] + [gc, c] + \text{tr} \{JQ, c[Q, c]\}
\]

and

\[
\text{tr} \{JQ, c[Q, c]\} = [Q, c] - 1/4|g, c|^2 + g^T, Qc.
\]

Hence, if \(c = y\gamma\), then

\[
|Q + gc, c| = [Q, c] + [gc, c] + \text{tr} \{JQ, c[Q, c]\}
\]

and the matrix is positive definite for large enough \(\gamma > 0\).

Corollary 1.1: If \(D\) is focal, then there exists a \(c \in \mathbb{R}^2\) such that \([JD], [D'Jb, J]\), agree in sign if and only if \(|D, x| = |AD^{-1}b, b| > 0\).

Proof: Assume \([JD], > 0\). According to Lemma 2, \([D'Jb, J], > 0\) iff \([D'Jb, J], > 0\) if and only if \([D', Jb, J], > 0\) if and only if \([D', Jb, J], > 0\). But

\[
[D'Jb, J], > 0 \iff [D', Jb, J], > 0
\]

Hence, \([D', Jb, J], > 0\) iff \(|D, x| = |AD^{-1}b, b| > 0\) since \(|D| > 0\).

The identical proof holds for \([JD], < 0\).

Corollary 2.2: If \(D\) is focal and \(|D, x| = |AD^{-1}b, b| > 0\), then if \(\gamma \in \mathbb{R}\), \(\gamma \neq 0\), \(D'Jb, J\), implies \([JD], [D'Jb, J]\), agree in sign when \(\gamma\) is large enough and \(\text{sgn} y = \text{sgn} x^T[JD], x\). In this case, \(D^{-1}A\), is \(D^{-1}-\)critical.

Proof: This follows directly from the construction of \(g\) in the proof of Lemma 2 when \(Q \neq D'Jb, J\).

Corollary 2.3: If \(D\) is focal and \(|AD^{-1}b, b| = 0 (b \neq 0)\), then \([JD], [D'Jb, J]\), agree in sign iff \(b \neq 0\) and \(y\gamma D'Jb, J\), where \(d\) is the other eigenvalue of \(A, \gamma\) in the sense of LTI pole-placement [8] and, thereby, of \((A, b, D)\) in our sense. The following proposition exploits this coincidence, specifying stabilizability conditions which make implicit use of this fact.

Proposition 1: If \(D\) is focal and \(|AD^{-1}b, b| \neq 0\), then there exists a \(c \in \mathbb{R}^2\) such that (3) is GAS if and only if \(|AD^{-1}b, b|, |D, x| > 0\) for all \(x \in \mathbb{R}^2\), we have \(|A, b| > 0\) when \(\gamma\) is large enough.

Since

\[
|A, b| = |A| + |b, c| + \text{tr} \{Jb, J\}
\]

we have \(|A, b| > 0\) when \(\gamma\) is large enough, since \(|D, x| > 0 (D\) is focal) and \(\gamma|AD^{-1}b, b| > 0\) since \(\text{sgn} y = \text{sgn} x^T[JD], x\).

But \(|A, b| \leq 0, |A, b| > 0\) implies \(A\), stable.

If \(D\) is focal, but \(b\) is an eigenvector of \(A, \gamma\), then according to Lemma 2 and its corollaries, \(D^{-1}A\), cannot be made focal by arbitrary choice of \(b\).

By Corollary 2.3, there exists a unique \(c \in \mathbb{R}^2\) such that conditions ii) and iii-a) of Theorem 2 hold; however, there is no guarantee that \(A, b\) is \(b\)-critical.
stabilizable in the sense of LTI pole-placement. Hence, the conditions for stabilizability in this case are more restrictive, and are given as follows.

**Proposition 2.** Let $D$ be focal, $|AD^{-1}b| > 0$ and let $d$ be the other eigenvector of $AD^{-1}$ with eigenvalue $\delta$. Then there exists a $c \in \mathbb{R}^2$ such that (3) is GAS if and only if either

i) $AD^{-1}$ is $b$-critical and $|A| > 0$ or

ii) $AD^{-1}$ is nodal and $|d| > 0$.

**Proof:** According to conditions i) and iii-b) of Theorem 2, $|A| > 0$ and $A: D = \gamma D$ hold. Therefore, $A: D = \gamma D$ and $|A| > 0$ imply $|d| > 0$. Hence, the conditions for stabilizability of (3) are satisfied.

In general, when $D = \delta D$, $d \notin (e_i)$, hence, $D$ is nodal as well as singular. In this case, stabilizability conditions are quite simple. The central result of this paper is the statement of necessary and sufficient conditions for stabilizability of (1) under constant linear state feedback as given by Theorem 1 in the Introduction. As a formal proof of this result we may summarize the results of Sections II–IV.

**IV. STABILIZABILITY WHEN $D$ IS SINGULAR**

If $D$ is nonsingular and not focal, then system (3) violates Theorem 2 as shown by Lemma 1. However, if $|D| = 0$, then for some $d, e \in \mathbb{R}^n$, $D = \delta D$. Hence, $c_x^T dx = c_x^T d_x$, and by choosing $c \in (e_i)$, $D'$ in system (2) is $d$-critical and singular (if $e \in (d')$), then system (2) is $d$-critical and singular to begin with). Therefore, when $D$ is singular, it is possible to stabilize (1) in some cases. Before presenting these cases, we state the following useful result.

**Lemma 3:** If $D$ is singular and $d$-critical and $|A| > 0$, then condition ii) of Theorem 2 holds iff $Ad = ad, a < 0$.

**Proof:** Since $x^T D' J A x = |A| x, D_x x = |A| x, A^{-1} d_x x^T x$ and $|d_x x|$, condition ii) is equivalent to

$0 < x^T D' J A x = |A| x, A^{-1} d_x x^T x = |A| x, A^{-1} d_x x^T x = |A| x, A^{-1} d_x x^T x$

which is possible, assuming $|A| > 0$, if and only if $0 < x, A^{-1} d_x x$.

In general, when $D = \delta D$, $d \notin (e_i)$, hence, $D$ is nodal as well as singular. In this case, stabilizability conditions are quite simple. The central result of this paper is the statement of necessary and sufficient conditions for stabilizability of (1) under constant linear state feedback as given by Theorem 1 in the Introduction. As a formal proof of this result we may summarize the results of Sections II–IV.

**V. SUMMARY AND CONCLUSION**

The central result of this paper is the statement of necessary and sufficient conditions for the stabilizability of (1) under constant linear state feedback as given by Theorem 1 in the Introduction. As a formal proof of this result we may summarize the results of Sections II–IV.

If $D$ is focal and $|AD^{-1}b| > 0$, then (3) is GAS iff $sgn(Ad) = sgn|dx, x|$, according to Proposition 1 (Section II). In this case, a stabilizing controller is given by $c \in D^{-1/2} \gamma D$, $sgn \gamma = sgn|dx, x|$, and $|\gamma|$ suitable large. If $|AD^{-1}b| = 0$ and $d$ is the eigenvector of $AD^{-1}$, then a stabilizing controller given by $c \in D^{-1/2} \gamma D$, $sgn \gamma = sgn|dx, x|$ may be chosen iff the conditions of Proposition 2 (Section III) hold. Thus, if $D$ is focal, condition i) of Theorem 1 is necessary and sufficient for stabilizability.

If $D$ is singular and nodal then (3) is GAS if $d$, its nonzero eigenvector, is a stable eigenvector of $A$, according to Proposition 3 (Section IV). In this case, $c \in D^{-1/2} \gamma D$, $sgn \gamma = sgn |d, b|$, and $|\gamma|$ suitable large is a stabilizing controller. If $D$ is singular and critical, then (3) is GAS iff the conditions of Proposition 4 (Section IV) hold. If $D$ is $b$-critical and those conditions are met, then any $e$ which stabilizes $(A, b)$ in the sense of LTI pole-placement, stabilizes (1). If $D$ is $d$-critical, $e \notin (d)$, and the conditions are met, then $c \in D^{-1/2} \gamma D$, $sgn \gamma = sgn |d, b|$ is a stabilizing controller. Thus, if $D$ is singular, condition ii) of Theorem 1 is necessary and sufficient for stabilizability.

If $D$ is neither focal nor singular, then (3) is never GAS, according to Lemma 1 (Section II). Thus, Theorem 1 lists complete necessary and sufficient conditions for stabilizability, as claimed.

**REFERENCES**


