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Abstract

This paper introduces a class of linearizing coordinate transformations for mechanical systems whose moment of inertia matrix has a square root which is a jacobian. The transformations, when they exist, define a local isometry from joint space to euclidean space, hence, may afford further insight into the transient behavior of robot motion. It remains to be seen whether any appreciably large class of robots admit such linearizing isometries.

1   Introduction

This paper will propose a smooth coordinate transformation which, when it exists, exactly linearizes the dynamical equations of a mechanical system with n degrees of freedom. The transformation is considerably simpler than well known exact linearization schemes based upon cancelling all nonlinear terms through actuator inputs. Moreover, it is defined on the entire phase space and gives rise to a local isometry between configuration space and euclidean n-space. The existence of this transformation depends upon the solution to a system of partial differential equations governed by the robot's kinematics along with its dynamical parameters. Thus, this paper raises but does not answer the question as to which robots behave isometrically like a system of linear time invariant double integrators with memoryless nonlinear input and output functions.

After introducing some terminology, below, a "feedback cancellation" coordinate transformation commonly encountered in the robotics literature is presented in Section 2. The new transformation is presented in Section 3.

A brief discussion of the difficulties involved in developing practical existence tests and construction techniques is provided in the concluding section. Definitions and notation are relegated to the appendix.

Consider the rigid body model of robot dynamics for an n degree of freedom kinematic chain,

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + k(q) = u, \tag{1}$$

where the generalized positions take values in *joint space*, $q \in J$, and there is an acuator for every degree of freedom, $u \in \mathcal{U} \subset \mathbf{R}^n$. For ease of exposition we will assume that J is a compact simply connected subset of $\mathbf{R}^n$ - a condition prevailing for all robots whose revolute joints are constrained to move over an arc less than 360°, and which may be relaxed, in any case, with more attention to technical details. Every robot of practical interest possesses a positive definite *moment of inertia matrix*, and we assume this true of M. Throughout the sequel, we will assume that the gravitational torques, k, are zero. For many kinematic designs, for instance "SCARA" arms, this is a realistic assumption. Otherwise, command inputs, u, discussed below must be augmented by a cancellation term,

$$u_{aug} \triangleq u + k(q),$$

to be meaningful.

We adopt the usual model of *workspace*, $\mathcal{W} \triangleq SO(3) \times \mathbf{R}^3$, and denote the *kinematics* $g : J \longrightarrow \mathcal{W}$. Equation (1) may be derived according to the Lagrangian formulation of Newton's laws. In so doing, it becomes clear that the particular form of M, and, therefore, B, is governed by the kinematics in conjunction with the robot's dynamical parameters, $p \in \mathbf{R}^{10n}$. We assume this derivation is familiar, and simply note for later use that

$$B\dot{q} = \dot{M}\dot{q} - \frac{1}{2}[d_q\dot{q}^{\mathrm{T}}M\dot{q}]^{\mathrm{T}}. \tag{2}$$

Since M is non-singular, system (1) is equivalent to

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -M^{-1}(x_1)[B(x_1,x_2)x_2 + k(x_1) - u] \\
y &= g(x_1)
\end{aligned}
\tag{3}
$$

where the generalized positions and velocities take values in *phase space*, $x \in \mathcal{P} \triangleq TJ$, with coordinates $x_1 \triangleq q; x_2 \triangleq \frac{d}{dt}q$.

2 Linearizations of Mechanical Systems

It is well known that rigid body dynamical models of robot arms may be linearized exactly using a suitable coordinate transformation, for instance, as follows.

Lemma 1 *Consider the rigid body model of a mechanical system, (3), in the absence of gravity, $k = 0$. Let $h : J \to \mathbf{R}^n$ be a local diffeomorphism. Then under the change of coordinates, defined by $T : \mathcal{P} \times \mathcal{U} \to \mathbf{R}^{3n}$,*

$$
\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = T(x_1,x_2,u) \triangleq \begin{bmatrix} h(x_1) \\ dh\, x_2 \\ \dot{dh}\, x_2 - dh\, M^{-1}[Bx_2 - u] \end{bmatrix}
\tag{4}
$$

the system has linear time invariant dynamics given by

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= v
\end{aligned}
\tag{5}
$$

with output map

$$
y = g \circ h^{-1}(z_1).
$$

Proof: According to the definition of z_1, z_2,

$$
\dot{z}_1 = dh\, x_2 = z_2,
$$

by applying the chain rule. Similarly,

$$
\dot{z}_2 = \dot{dh}\, x_2 - dh\, M^{-1}[Bx_2 + k - u].
$$

$\diamond$

This is not the most general class of transformations that might be used to linearize (3), according to the recent nonlinear systems literature, e.g. [2], but it includes methods commonly encountered in the field of robotics. In particular, for non-redundant kinematics, if we identify h, the first component of T with the kinematic map,

$$
h(x_1) \triangleq g(x_1),
$$

then, locally, T not only linearizes (3), but dynamically decouples each input and output pair, e.g., as reported in [3].

Evidently, the advantage attending this point of view, is the possibility of applying classical theory directly to the control of nonlinear plants. For instance, if K, Γ, respectively, define a linear feedback law and precompensating filter,

$$
v_r \triangleq -Kz + \Gamma(r),
\tag{6}
$$

under whose action the output, z_1, of (5) behaves in a desired fashion with respect to the reference input, r then the input to the robot (3) defined by the inverse coordinate transformation for u under T, (4),

$$
u_r \triangleq Bx_2 + M dg^{-1}[v_r - \dot{dg}\, x_2],
\tag{7}
$$

forces the output $y(t) \equiv z_1(t)$. Similar observations have been made independently by a variety of researchers in the field of robotics using different language. A well known example is provided by the "resolved acceleration" method of [4]: the control law, u_r is applied to the robot with the choice of Γ given as

$$
\Gamma(r) \triangleq K \begin{bmatrix} r \\ \dot{r} \end{bmatrix} + \ddot{r},
$$

the inverse of the filter specified by the equivalent closed loop linear time invariant system (5).

Note that this choice for h satisfies the hypothesis of the lemma almost everywhere in J, but will not define a viable transformation at the "kinematic singularities",

$$
\mathcal{C} \triangleq \{q \in J : rank(dg) < dim\mathcal{W}\},
$$

the critical points of g. Most realistic robots have kinematic singularities whose image under g is in the interior of $\mathcal{W}$ and which may not be easily located, hence such a transformation may be impracticable.

It is apparent from (7) that the success of this procedure is based upon exact cancellation of each term in B, as well as the ability to match every term in M. [1] In all likelihood, such a control methodology will be very sensitive to inaccuracies in the original model, (1), uncertainties in the available estimates of system parameters, as well as computational error arising from the digital implmentation of the control law. It is reasonable to inquire regarding the existence of simpler linearization schemes.

3 A Linearizing Isometry

The contribution of this paper is the observation that if the kinematics and dynamical parameters which give rise to system (1) define a moment of inertia matrix whose square root is the jacobian of some coordinate transformation then the previous inquiry may be answered in the affirmative. For ease of discussion we define the set of *square roots* of a smooth positive definite symmetric matrix valued function, $M(q)$, as

$$
\mathcal{N}(M) \triangleq \{N \in C^\infty[J, \mathbf{R}^{n \times n}] : NN^\mathsf{T} = M\}
$$

Theorem 1 *Consider the rigid body model of a mechanical system, (1), in the absence of gravity, $k(q) = 0$. Suppose there exists a smooth map, $h : J \to \mathbf{R}^n$, such that $dh^\mathsf{T} = N \in \mathcal{N}(M)$. Then under the change of coordinates, defined by $T : \mathcal{P} \times \mathcal{U} \to \mathbf{R}^{3n}$,*

$$
\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = T(x_1,x_2,u) \triangleq \begin{bmatrix} h(x_1) \\ N^\mathsf{T}x_2 \\ N^{-1}u \end{bmatrix},
\tag{8}
$$

the system has linear time invariant dynamics given by (5).

Proof: First note that since M is assumed to be positive definite, $\mathcal{N}(M)$ is not empty, and any h satisfying the hypothesis is an immersion. By construction, we have $z_2 \triangleq dh\, x_2 = \dot{z}_1$. Moreover,

$$
\begin{aligned}
\dot{z}_2 &= \dot{dh}\, x_2 + dh\, \dot{x}_2 \\
&= \dot{N^\mathsf{T}}x_2 - N^\mathsf{T}[NN^\mathsf{T}]^{-1}[Bx_2 - u] \quad \text{from (3)} \\
&= [\dot{N^\mathsf{T}} - N^{-1}B]x_2 + N^{-1}u,
\end{aligned}
$$

---

[1]And, in the presence of gravity, cancellation of k as well via u_{aug} as dicussed in the introduction.

and it remains to show that $[\dot{N}^\mathsf{T} - N^{-1}B] = 0$. To see this, recall, from equation (2),

$$Bx_2 \triangleq \dot{M}x_2 - \tfrac{1}{2}[d_q z_2^\mathsf{T} z_2]^\mathsf{T}$$

$$= [\dot{N}N^\mathsf{T} + N\dot{N}^\mathsf{T}]x_2 - [d_q \tfrac{d}{dt} z_1]^\mathsf{T} z_2$$

$$= [\dot{N}N^\mathsf{T} + N\dot{N}^\mathsf{T}]x_2 - [\tfrac{d}{dt}d_q z_1]^\mathsf{T} z_2$$

$$= N\dot{N}^\mathsf{T}x_2,$$

from which the result follows. Note that the exchanged order of differentiation in the third line is justified since z_1 is continuously differentiable in both q and t.

<div align="right">◇</div>

Corollary 1 *If it exists, the map, h, defined in Theorem 1 is a local isometry from $(J, \langle \cdot \mid \cdot \rangle_M)$ to $(\mathbf{R}^n, \langle \cdot \mid \cdot \rangle_I)$.*

Proof: Let $w_z = dh\, v_q$, $w_z' = dh\, v_q'$ be two tangent vectors in $T_z \mathbf{R}^n$, at $z = h(q)$, the respective images of tangent vectors $v_q, v_q' \in T_q J$, under the differential of h. We must show that their inner products are identical, namely

$$\langle v_q \mid v_q' \rangle_M = v_q^\mathsf{T} M v_q'$$

$$= v_q^\mathsf{T} N N^\mathsf{T} v_q'$$

$$= dh\, v_q^\mathsf{T} dh\, v_q'$$

$$= \langle w_z \mid w_z' \rangle_I,$$

as required. <div align="right">◇</div>

Some of the advantages of a coordinate transformation based upon the square root of the moment of inertia matrix are immediately evident. Given the choice of classical controller, v_r, from equation (6), the inverse transformation for u in terms of z, v is considerably simplified,

$$u_r \triangleq dh^\mathsf{T} v_r$$

in comparison to (7). Moreover, since h is an immersion, T may be computed everywhere on J. Finally, from the point of view of sensitivity raised at the end of the previous section, there is likely to be some advantage gained in not attempting the cancellation of B (which is quadratic in $\dot{q}$) via feedback. [2]

The existence of a local isometry between joint space with its inertia metric, and ordinary euclidean space would imply a close relationship between the motions of systems (3) and (5) which is bound to have important implications for the analysis of robot transient response.

4 Which Robots Possess Linearizing Isometries?

It seems ill-advised to pursue the pragmatic implications of these observations until it becomes clear that the trans-

formation is not a mere chimera. Conditions for the existence of a smooth map whose jacobian is in $\mathcal{N}(M)$ are identical to conditions for the existence of solutions to a system of $\frac{n(n+1)}{2}$ partial differential equations in n variables defined by the entries of M. An important and unresolved question, then, concerns the class of kinematic chains for which there exist dynamical parameters defining a moment of inertia matrix admitting solutions to these equations.

Even when such a transformation exists, it may not be easy to evaluate. Consider the one degree of freedom "mechanical system" built from kinetic energy defined by

$$K \triangleq \frac{1}{2}m(q)\dot{q}_2^2$$

(not necessarily corresponding to any physical system) where m is a positive scalar function,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \tfrac{1}{m}[-\tfrac{1}{2}\dot{m}x_2 + u].$$

An isometry defined by

$$h(x_1) \triangleq \int_0^{x_1} m(\varsigma)^{\frac{1}{2}} d\varsigma$$

always exists for this system, but it is easy to choose functions, e.g. $m \triangleq 1 + \cos^2 q$, for which h appears to admit no closed form expression in terms of elementary functions. Such transformations might have more analytical utility than practical function in an on-line control setting.

Despite these serious unresolved questions, the potential value of linearizing isometries appears intriguing. A more definitive account of the problems introduced here is the topic of a future paper.

Appendix

The *differential* of a smooth function, f, will be denoted df, which will be used to denote its *jacobian* matrix representation as well. For functions, $f(x, y)$, we denote the "partial" differentials

$$d_x f \triangleq df \begin{bmatrix} I \\ 0 \end{bmatrix} \quad d_y f \triangleq df \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Denoting the vector field in (3) as $X(x, u)$, there should be no confusion introduced by denoting the Lie derivative of any smooth map, h, along the flow of X as

$$\dot{h} \triangleq L_X(h),$$

or, occasionally, $\frac{d}{dt}h$.

A *Riemannian Metric* is a positive definite quadratic form defined on the tangent space at each point of a smooth manifold in such a fashion that the entries of any matrix representation are smooth scalar maps on the manifold. Every such quadratic form defines an *inner product*, $\langle \cdot \mid \cdot \rangle$ defined for all tangent vectors. [3] A *Riemannian Manifold* is a manifold, $\mathcal{R}$, possessed of a Riemannian Metric $\langle \cdot \mid \cdot \rangle$. Thus, $(J, \langle \cdot \mid \cdot \rangle_M)$ is a Riemannian Manifold when we define

$$\langle v_q \mid v_q' \rangle_M \triangleq v_q^\mathsf{T} M(q) v_q'$$

for all $q \in J$ and $v_q, v_q' \in T_q J$. A *local isometry* is a smooth map between two Riemannian Manifolds

$$h : (\mathcal{R}_1, \langle \cdot \mid \cdot \rangle_1) \longrightarrow (\mathcal{R}_2, \langle \cdot \mid \cdot \rangle_2)$$

---

[2] Of course, dh depends upon the uncertain dynamical parameters, and can be no more accurately computed than M, as in (7). Again, in the presence of gravitational disturbances, u_r will contain an extra term as well.

[3] A simple reference for the definitions introduced in this paragraph is the book by Thorpe [1].

which is a local diffeomorphism and preserves inner products, i.e. for all $q \in \mathcal{R}_1$ and $v_q, v'_q \in T_q \mathcal{R}_1$ if

$$
\begin{aligned}
z &= h(q) \\
w_z &= dh \; v_q \\
w'_z &= dh \; v'_q
\end{aligned}
$$

then

$$
\langle v_m \mid v'_m \rangle_1 = \langle w_z \mid w'_z \rangle_2.
$$

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