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Abstract

This paper presents a strict, global Lyapunov function for the class of dissipative mechanical systems defined on a configuration space which admits a trivial tangent bundle.

1 Introduction

The "mechanical systems" define a large and important class of highly nonlinear dynamical equations. For example, all robot dynamics are modeled by such systems. Since there can be no "operational calculus", Lyapunov theory offers an attractive alternative approach to a constructive control methodology for this class of systems: it is not hard to translate many frequency domain insights into this language in the linear case. [8]. Generalizing such an approach obviously depends upon the availability of "adequate" Lyapunov functions.

Here it seems on first impression as though nature has been kind. There is a "canonical" Lyapunov function for dissipative mechanical systems — the total energy — as has been known for since the nineteenth century. This scalar valued map enjoys the property of global definition, since it is constructed in terms of the intrinsic Riemannian geometry of the configuration space. Unfortunately, it results in a semi-definite derivative along trajectories of a mechanical system, and the full strength of Lyapunov's method is lost.

Recently, Arimoto [2], Bayard and Wen [15], and the present author [5,6] have, all independently, devised strict Lyapunov functions for "PD" compensated mechanical systems. All three constructions are remarkably similar, and share the common fault of achieving only a locally negative definite derivative (the local domain may be arbitrarily enlarged) on the phase space. The central contribution of this paper, Theorem 1, is the construction of a new Lyapunov function for the dissipative mechanical systems which is *global* — defined in terms of the intrinsic Riemannian geometry of an arbitrary mechanical system — and *strict* — the derivative is negative definite on the phase space of (tangent bundle over) any parallelizable configuration space, vanishing only on the equilibrium states of the vector field.¹

This paper, being of an essentially analytical nature, will make only illustrative use of the new construction to re-derive (hopefully in a simpler form) the useful results concerning PD compensated (quadratic potential) mechanical systems on the "Euclidean n -disk" reported in the conference papers cited previously [2,15,6]. Namely, such systems are exponentially stable, hence BIBO stable, and steady state output magnitudes may be shown to be proportional to input magnitudes with a constant of proportionality depending in a simple fashion upon the PD gain magnitudes. The larger motivation for this work is the hope of applying the construction to more general robotic problems where the configuration space is not homeomorphic to a disk and the potential energy is not quadratic. This work is currently in preparation.

¹The limitation to parallelizable spaces — i.e., those whose tangent bundle is trivial — seems to be an artifact of the present proof rather than an intrinsic limitation in the construction. This issue is presently under investigation.

The next section presents the ingredients leading to the standard definition of a dissipative mechanical system on a smooth configuration space. Section 3 introduces the new Lyapunov function. Finally, Section 4 treats the PD compensated mechanical system, as an example of the construction in the familiar case. A variety of technical results and computations are not included in this paper for lack of space — the reader is referred to the background technical report [4] for further details.

2 Dissipative Mechanical Systems

The geometry of classical physics has been extensively studied for decades, and recent years have witnessed the publication of numerous expository texts containing the background material required for the present paper. This section sketches the relevant ideas, appealing frequently to the excellent text of Abraham and Marsden [1].

2.1 Notation

The tangent bundle over a manifold, J , will be denoted TJ with projection map $\tau : TJ \rightarrow J$. The coordinate representation of a map will be denoted by underlining. For the purposes of this paper, the manifold, J , will be a simply connected open subset of \mathbb{R}^n , which admits a single global chart, and, when there is no chance of confusion, coordinate representations of maps will often be denoted by the same symbol as the original map. Taking the jacobian of a map between Euclidean Vector spaces is denoted by D , in contrast to taking the differential one form of a scalar valued map on an arbitrary manifold, d .

Following Hirsch [3], if X, Y are two vector bundles over J , then define the *morphisms* between them, $\mathcal{M}^r[X, Y]$, to be the set of all smooth (i.e. continuously differentiable r times) fiber preserving maps which are linear on each fiber, i.e.,

$$\mathcal{M}^r[X, Y] \triangleq \{M \in C^r[X, Y] : \\ M | \pi_X^{-1}(q) \in \mathcal{L}(\pi_X^{-1}(q), \pi_Y^{-1}(q)) \text{ for all } q \in J\}.$$

Transpose, symmetric, and positive definite morphisms are presented in [4], along with the notion of upper and lower magnitude functions, and the group of scalar comparison functions,

$$\mathcal{K}_r \triangleq \{\kappa \in C^r[\mathbb{R}^+, \mathbb{R}^+] : \\ \lim_{t \rightarrow 0} \kappa(t) = 0 \text{ and } \lim_{t \rightarrow \infty} \kappa(t) = \infty \\ \text{and } t_1 < t_2 \Rightarrow \kappa(t_1) < \kappa(t_2)\},$$

which are used to define Lyapunov functions on manifolds.

2.2 Mechanical Systems

2.2.1 Kinetic Energy as a Riemannian Metric

Define a *mechanical system* to be the Lagrangian dynamical system resulting from a cost functional specified by a Rie-

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mannian metric — a positive definite symmetric morphism $M \in \mathcal{M}^\omega[TJ, T^*J]$, as discussed in [4] — which specifies an inner product $\langle \cdot | \cdot \rangle$ on T_qJ for every point of configuration space, $q \in J$. The author has presented a quick derivation of the Riemannian metric resulting from the sort of physical system which motivates the paper — the *kinetic energy* ², $\kappa(v) \triangleq \frac{1}{2} \langle v | v \rangle$ of a “kinematic chain” encountered in robotics — in a recent encyclopedia article [8].

A Riemannian metric induces an *affine connection*, ∇ , with respect to which *parallel translation* preserves the metric inner product [1]. This implies that for any two curves, c_1, c_2 , in TJ over the same base curve, $b \equiv \tau \circ c_1 \equiv \tau \circ c_2$, we have

$$\frac{d}{dt} \langle c_1 | c_2 \rangle = \langle \nabla_{\dot{c}_1} c_1 | c_2 \rangle + \langle c_1 | \nabla_{\dot{c}_2} c_2 \rangle,$$

taking the *covariant derivative*, along the curves c_i , considered as vector fields over the base curve, b [9].

2.2.2 Coordinate Representation of the Lagrangian Vector Field

Each point, $q \in J$, admits an open neighborhood, $O \subseteq J$, on which the “local phase space at q ”, is a Euclidean vector space. Thus, a point, $v \in TJ$ admits a local coordinate representation,

$$p \triangleq \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{R}^{2n}.$$

For the purposes of the present paper, the configuration space may be presumed a subset of \mathbb{R}^n , thus, TJ is trivial — i.e. it is a subset of $\mathbb{R}^n \times \mathbb{R}^n$ — and this local coordinate system is valid on the entire phase space, TJ .

The equations of motion in the local coordinates resulting from application of the “Euler-Lagrange” operator, to the kinetic energy, κ , take the form

$$\dot{p}_1 + \Gamma_{p_1}(\dot{p}_1, \dot{p}_1) = 0,$$

where Γ represents the *Christoffel symbols* for the given coordinates. The relation of Γ to the coordinate expression of the morphism, \underline{M} , is derived in [4] as

$$\begin{aligned} \Gamma_{p_1}(x, y) &= \frac{1}{2} \underline{M}^{-1}(p_1) C(p_1, x) y \\ C(p_1, x) y &\triangleq \left[\underline{M}_{p_1}(x) y + \underline{M}_{p_1}(y) x - \underline{M}_{p_1}(x)^T y \right] \\ \underline{M}_{p_1}(x) &\triangleq (x \otimes I)^T D \underline{M}^S, \end{aligned}$$

where it is also shown that Γ is symmetric — $\Gamma(x, y) = \Gamma(y, x)$ — so that this is the expression of the unique torsion free metric connection. ³ This results in the local representation of f_κ as

$$\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -\underline{M}^{-1}(p_1) C(p_1, p_2) p_2. \end{aligned} \quad (1)$$

While TJ is endowed with a natural metric induced by the Riemannian metric, M , as described in [4], the present paper will require only a local notion of distance, for purposes of applying comparison functions to TJ [4]. In such applications

²Kinetic Energy is formally defined as a scalar valued map on T^*J [1][Def. 4.5.2], however it will ease the discussion and do no technical harm in the present paper to speak of κ as a map on TJ .

³In earlier papers, [7,6], the author has unwittingly used the local expression for a metric connection with non-zero torsion. While the geodesics of two connections which differ only by torsion are identical, the non-zero torsion necessitates annoying additional cancellation of “defect” terms in control applications. It is worth pointing out that other authors have independently made use of the preferable zero-torsion connection in their control applications [15,12].

it suffices to use the Euclidean norm topology induced by the metric of the local coordinate system. A metric, ρ , having been chosen, any morphism, F , gives rise to scalar valued maps on J , $\nu_F(q), \mu_F(q)$, as introduced in [4], which measure its smallest and greatest “magnitudes” on each fiber, and lead to an obvious construction of a quadratic comparison function on each fiber.

Lemma 1 For every $q \in J$, the kinetic energy, κ , is a non-degenerate positive definite function on T_qJ , possessed of a quadratic lower comparison function, in coordinates,

$$\kappa(p_1, p_2) \geq \overline{\nu_M} \|p_2\|^2,$$

2.3 Gradient Vector Fields in Mechanical Systems

2.3.1 Morse Functions and Their Gradient Vector Fields

Consider the class of twice differentiable real valued functions $\varphi \in C^2[J, \mathbb{R}]$. The *co-vector field*, $d\varphi$, is related to the *gradient vector field*, $\text{grad } \varphi$, of φ by the identity

$$\text{grad } \varphi \triangleq M^{-1} d\varphi.$$

Note that the “hessian” of the original function is obtained from the jacobian, $\tilde{K}_1 \triangleq d\varphi_*$, in local coordinates,

$$\tilde{K}_1(p_1) \equiv [D^2 \varphi](p_1). \quad (2)$$

One calls φ a *Morse function* if its hessian is non-singular at every critical point [3].

2.3.2 Lifting Gradient Vector Fields

Any smooth real valued function, $\varphi : J \rightarrow \mathbb{R}$ may be “pulled back” to TJ in a natural fashion by defining $\tilde{\varphi} \triangleq \varphi \circ \tau$. This defines a new Lagrangian function,

$$\lambda \triangleq \kappa - \tilde{\varphi}, \quad (3)$$

whose Lagrangian vector field, $f_{\kappa-\tilde{\varphi}}$, can be shown to include a “lift” of the gradient vector field of φ in the following sense. It can be shown [1] that c is a trajectory of $f_{\kappa-\tilde{\varphi}}$ if and only if its base integral curve, $b \triangleq \tau \circ c$ has the property

$$\nabla_{\dot{b}} \dot{b} = -\text{grad } \varphi \circ b. \quad (4)$$

This implies that the resulting Lagrangian vector field may be written as

$$f_{\kappa-\tilde{\varphi}}(x) = f_\kappa(x) - V_x(\text{grad } \varphi) \circ \tau(x),$$

where V is the vertical lift.

2.3.3 Coordinate Representation of a Mechanical System with Good Potential Energy

Say that $\varphi \in C^2[J, \overline{\mathbb{R}^+}]$ is a *good potential function* at q_0 if it is a Morse function on J , positive definite at q_0 , and bounded away from zero everywhere else on J . This means that there can be found a lower comparison function, $\lambda_\varphi \in \mathcal{K}_2$, for φ , in the sense defined in [4].

$$\lambda_\varphi \circ \rho_J(q_0, q) \leq \varphi(q) \quad (5)$$

In such a case, q_0 is a local minimum of φ — an attracting equilibrium state of $-\text{grad } \varphi$. It follows that ⁴ $\|d\varphi\|$ is a

⁴The metric, M , on TJ induces a metric on T^*J , with respect to which the norm of a co-vector is well defined.

decreascent positive definite function at q_0 , admitting the upper comparison, $v_{d\varphi} \in K_1$, with the property

$$v_{d\varphi} \circ \rho_J(q_0, q) \geq \|d\varphi\|. \quad (6)$$

Using the same arguments as in [3][Lem.6.1.1], given any local coordinate system on a neighborhood, \mathcal{O} , of $q \in J$, φ induces a smooth map $\underline{K}_1 : \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\underline{M}^{-1}(p_1)\underline{K}_1(p_1)p_1$ is a representation of $\text{grad } \varphi$. It now follows that the vector field, $f_{\kappa-\varphi}$ admits the coordinate representation

$$\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -\underline{M}^{-1}(p_1)(C(p_1, p_2)p_2 + \underline{K}_1(p_1)p_1) \end{aligned} \quad (7)$$

2.4 Strict Dissipative Vector Fields and External Forces

Following Abraham and Marsden, once again, say that a smooth vector field, f_d , on TJ is *dissipative* [1] if it is vertical and has the property with respect to the kinetic energy, κ , that

$$d\kappa f_d \leq 0.$$

Since f_d is vertical, it may be uniquely associated with a smooth fiber map, $k_2 : TJ \rightarrow T^*J$, via lifting

$$f_d(v) \triangleq -V_v \circ M^{-1}k_2(v).$$

If the product $k_2(v) \cdot v$ is a positive definite map on T_qJ for every $q \in J$, then say that f_d is a *strict dissipative vector field*. In this case, there may be found a smooth map, $K_2 : T_qJ \rightarrow \mathcal{L}(T_qJ, T_q^*J)$ such that $k_2(v) = K_2(v)v$, and $(K_2)_s$, the symmetric part of K_2 as defined in [4] is always positive definite.

Lemma 2 *If f_d is a strict dissipative vector field on TJ , then $-d\kappa f_d$ is a positive definite function on T_qJ with a quadratic lower comparison function, in coordinates,*

$$k_2(p_1, p_2) \cdot p_2 \geq \overline{\nu K_2} \|p_2\|^2.$$

2.5 Dissipative Mechanical Systems

To summarize, a kinetic energy law, κ , determined by a Riemannian metric, M , on J , induces a mechanical system (1) through the Lagrangian vector field, f_κ , whose base integral curves are geodesic with respect to M . The addition of a potential function, φ results in a new Lagrangian vector field,

$$f_{\kappa-\varphi} = f_\kappa - V(\text{grad } \varphi) \circ \tau,$$

whose integral curves conserve total energy, $\eta = \kappa + \varphi$. Finally, the choice of a dissipative vector field, f_d , completes the list of ingredients. We will limit attention to following class of mechanical systems:

Definition:

A dissipative mechanical system is defined by the vector field

$$f \triangleq f_{\kappa-\varphi} + f_d, \quad (8)$$

where

1. κ is the kinetic energy induced by a Riemannian metric, M , on J ;
2. φ is a "good potential function" in the sense of Section 2.3.3;
3. f_d is a "strict" dissipative vector field in the sense of Section 2.4.

In local coordinates, the disturbed dissipative mechanical system can be written,

$$\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -\underline{M}^{-1}(p_1)(C(p_1, p_2) + K_2(p_1, p_2))p_2 + \underline{K}_1(p_1)p_1 \end{aligned} \quad (9)$$

3 The Stability of Dissipative Mechanical Systems

Lord Kelvin showed that total energy, $\eta = \kappa + \varphi$, is a Lyapunov function for a dissipative mechanical system in 1886 [14]. Arimoto and colleagues [10,13] contributed a precise demonstration that every minimum of the potential field is a local attractor by application of LaSalle's Invariance Principle. Similar independent work of Van der Schaft [11] and this author [7] appeared subsequently. Of course, the central idea of energy dissipation is to be found in the standard texts as well, [1][Prop. 3.7.17]. Since η is not a strict Lyapunov function — that is, the Lie derivative, $L_f(\eta)$ is negative semi-definite — conclusions about asymptotic properties of the flow near $(q_0, 0)$ require the application of LaSalle's Invariance Principle. This limits application to autonomous dynamics, precluding any input/output analysis of the effects of the external disturbance. This flaw will now be corrected: we will "fix" Lord Kelvin's formula by adding a "cross term" to the total energy function which makes its derivative along the motions negative definite.

3.1 The "Potential Angle" and Energy Scaling Function

The concern of this section, is the construction of a new scalar valued map on TJ whose co-vector field acts on the portion of f which is "ignored" by $d\eta$ — the "vertical lift" of the potential gradient vector field. Namely, a map, $\alpha \in C^1[TJ, \mathbb{R}]$, is constructed to measure the "angle" between the desired velocity of the lifted gradient flow, and the actual velocity of the dissipative system.

The Riemannian metric is a symmetric isomorphism in $\mathcal{M}^\omega[TJ, T^*J]$. By choosing a morphism in the other direction one obtains the desired angular comparison, as follows. Say that a morphism, $F \in \mathcal{M}^\omega[T^*J, TJ]$, is a *pre-metric* if its symmetric part is a "positive definite" operator on each fiber. Given a manifold, J , with a Riemannian metric, M , on the tangent bundle, $\tau : TJ \rightarrow J$, a pre-metric, F , and a scalar valued map, $\varphi \in C^1[J, \mathbb{R}]$, define the *potential angle map*, $\alpha \in C^1[TJ, \mathbb{R}]$ by

$$\alpha(v) \triangleq \langle Fd\varphi \circ \tau(v) \mid v \rangle_M. \quad (10)$$

Lemma 3 *The Lie derivative of the potential angle map, α , along the dissipative mechanical system, f , (9), is a scalar valued map on TJ which may be expressed as*

$$L_f\alpha(v) = \langle Bv \mid v \rangle - \langle Fd\varphi \circ \tau(v) \mid k_2(v) \rangle - (d\varphi \cdot Fd\varphi) \circ \tau(v),$$

where $B \in \mathcal{M}^\omega[TJ, TJ]$.

Proof: If c is any smooth curve in TJ , then it may be considered as vector field along the base curve, $\tau \circ c \triangleq b$. Thus,

$$\frac{d}{dt} \alpha \circ c = \langle \nabla_i Fd\varphi \circ b \mid c \rangle + \langle Fd\varphi \circ b \mid \nabla_i c \rangle,$$

since ∇ is the Riemannian connection. If c is the trajectory of any second order system then

$$\dot{b} = T\tau \circ \dot{c} = c,$$

hence

$$\begin{aligned} \frac{d}{dt} \alpha \circ c &= \langle \nabla_c Fd\varphi \circ b \mid c \rangle + \langle Fd\varphi \circ b \mid \nabla_{\dot{c}} \dot{b} \rangle \\ &= \langle Bc \mid c \rangle + \langle Fd\varphi \circ b \mid \nabla_{\dot{c}} \dot{b} \rangle, \end{aligned}$$

where $B \in \mathcal{M}^\omega[TJ, TJ]$ denotes the morphism $\nabla Fd\varphi$. Since b is the base integral curve of a dissipative mechanical system it follows that

$$\nabla_{\dot{c}} \dot{b} = -grad \varphi \circ b - k_2(c),$$

hence

$$\begin{aligned} \frac{d}{dt} \alpha \circ c &= \langle Bc \mid c \rangle - \langle Fd\varphi \circ b \mid k_2(c) \rangle \\ &\quad - \langle Fd\varphi \circ b \mid grad \varphi \circ b \rangle \\ &= \langle Bc \mid c \rangle - \langle Fd\varphi \circ b \mid k_2(c) \rangle \\ &\quad - (d\varphi \cdot Fd\varphi) \circ b, \end{aligned}$$

and the result follows

□

The pre-metric property of F insures that the last term of $L_f(\alpha)$ is globally non-positive, vanishing only on the critical points of φ , since F_\bullet defines a positive definite quadratic form on the co-tangent bundle. Thus, adding α to η introduces a negative term in the derivative which does not vanish automatically on the zero section of TJ . Unfortunately, α , itself, is sign indefinite, and its derivative along trajectories of f incurs two additional sign indefinite terms as well as computed by Lemma 3. It will now be shown that a "rescaling" of η obtained by composition with a suitable comparison function, $\gamma \in \mathcal{K}_1$, serves simultaneously to dominate the indefinite terms of both α and its derivative. Specifically, define

$$\begin{aligned} \gamma(\chi) &\triangleq \gamma_0\chi + \gamma_1(v_{d\varphi}^2 \circ \lambda_\varphi^{-1})(\chi) \\ &\quad + \gamma_2 \left[\int_0^\chi (v_{(MB)_\bullet} \circ \lambda_\eta^{-1})(\sigma) d\sigma \right], \end{aligned} \quad (11)$$

where $\gamma_0, \gamma_1, \gamma_2$ are positive non-decreasing scalar functions to be chosen below, and $v_{(MB)_\bullet}$ is a smooth non-decreasing function which satisfies the inequality

$$v_{(MB)_\bullet}(\chi) \geq \sup_{\rho_j(q_0, \dot{q}) \leq \chi} \mu_{(MB)_\bullet}(q),$$

where $\mu_{(MB)_\bullet}$ denotes the "tensor magnitude" of the symmetric part of the morphism MB on each fiber, as defined in [4], and $v_{d\varphi}$ is an upper comparison function for $\|d\varphi\|$ as defined in Section 2.3.3.

3.2 Verification of the Construction

First observe that γ preserves the sign definite properties of η when their composition is added to α .

Proposition 1 For every dissipative mechanical system, f , (8) and potential angle map, α , (10), there can be found a comparison function, $\gamma \in \mathcal{K}_1$, of the form (11), such that

$$\vartheta \triangleq \gamma \circ \eta + \alpha$$

is a positive definite function in $C^1[TJ, \overline{\mathbb{R}^+}]$.

Proof: Note that ϑ vanishes at $v_0 = (q_0, 0)$ as required. We first seek to show that ϑ is positive definite. Note that, in coordinates,

$$\eta \geq \tilde{\eta} \triangleq \lambda_\varphi(\|p_1\|) + \overline{v_M}\|p_2\|^2,$$

according to (5). Moreover,

$$\gamma(\chi) \geq \tilde{\gamma}(\chi) \triangleq \gamma_0\chi + \gamma_1(v_{d\varphi}^2 \circ \lambda_\varphi^{-1})(\chi),$$

from the definition of γ and (6) — it follows that

$$\gamma \circ \eta \geq \tilde{\gamma} \circ \tilde{\eta} \geq \gamma_0\overline{v_M}\|p_2\|^2 + \gamma_1 v_{d\varphi}^2(\|p_1\|).$$

Notice, moreover, in coordinates, that

$$|\alpha| \leq |D\varphi(p_1)E^T M p_2| \leq v_{d\varphi}(\|p_1\|) \cdot \mu_{F^T M}(p_1) \cdot \|p_2\|,$$

where $\mu_{F^T M}(p_1)$ is again the tensor magnitude function defined in [4]. It follows that

$$\vartheta \geq \gamma_1 v_{d\varphi}^2 - \mu_{F^T M} v_{d\varphi} \|p_2\| + \gamma_0 \overline{v_M} \|p_2\|^2$$

and it suffices⁵ to find non-decreasing functions, γ_0, γ_1 , such that $\gamma_i \circ \eta$ is smooth and

$$\begin{bmatrix} \gamma_1(\eta) & \frac{1}{2} \mu_{F^T M}(p_1) \\ \frac{1}{2} \mu_{F^T M}(p_1) & \gamma_0(\eta) v_M(p_1) \end{bmatrix} > 0.$$

For this, the conditions

$$\gamma_0 \geq 1; \quad \gamma_1 \geq \frac{1}{4\overline{v_M}} v_{F^T M} \circ \lambda_\eta^{-1}, \quad (12)$$

are sufficient, where $v_{F^T M}$ is a smooth non-decreasing function satisfying the inequality

□

The addition of the "angle measurement" in ϑ results in a locally negative definite derivative, $\dot{\vartheta}$. In fact, the appropriate rescaling of η affords dominance of the negative definite terms "as globally as the potential function can allow". This is made precise by the following central result.

Theorem 1 Let f be a dissipative mechanical system. Then for every valid potential angle map, α , (10), and a comparison function, $\gamma \in \mathcal{K}_1$, of the form (11), the positive definite function,

$$\vartheta \triangleq \gamma \circ \eta + \alpha$$

has a derivative along trajectories of f ,

$$\dot{\vartheta} = L_f(\vartheta),$$

which is non-positive on TJ and vanishes only on the equilibrium states of f .

Proof: Let $\gamma \in \mathcal{K}_1$ be chosen as in (11), so that ϑ is positive definite in consequence of Proposition 1. We have, along any curve, c , in TJ ,

$$\dot{\vartheta} = (\gamma' \circ \eta)\dot{\eta} + \dot{\alpha}.$$

If c is a trajectory of f , then, in coordinates,

$$\dot{\eta} = -p_2^T \overline{K}_2(p_1, p_2) p_2.$$

⁵The sufficiency depends upon the fact that $v_{d\varphi}$ is an upper comparison function for $\|d\varphi\|$, hence (in contrast, for instance, to a lower comparison function) may be guaranteed not to vanish except at the point q_0 .

Now γ' , the derivative of γ , satisfies

$$\gamma'(\chi) \geq \gamma_0 + \gamma_2 v_{(MB)_*} \circ \lambda_\eta^{-1}(\chi),$$

since all constituent functions of γ are non-decreasing. From $\lambda_\eta^{-1} \circ \eta \geq \|p\|$, it follows that

$$v_{(MB)_*} \circ \lambda_\eta^{-1} \circ \eta \geq v_{(MB)_*}(\|p\|) \geq v_{(MB)_*}(\|p_1\|)$$

hence,

$$\dot{\theta} \leq -p_2^T K_2 p_2 (\gamma_0 + \gamma_2 \mu_{(MB)_*}) + \dot{\alpha}.$$

Using the results of Lemma 3, expressed in coordinates, we have

$$\begin{aligned} \dot{\alpha} &= -D\varphi F^T D\varphi^T - D\varphi F^T M K_2 p_2 + p_2^T B^T M p_2 \\ &\leq -\overline{\nu_{F_*}} \|D\varphi\|^2 + \mu_{F^T M K_2} \|p_2\| \cdot \|D\varphi\| + \mu_{(MB)_*} \|p_2\|^2, \end{aligned}$$

where

$$\overline{\nu_{F_*}} \triangleq \inf_{q \in J} \nu_{F_*}(q)$$

is a positive constant according to the assumption that F is a pre-metric. It now follows that

$$\dot{\theta} \leq -l(p)^T Q(p) l(p) + v_{(MB)_*}(\|p_1\|) (1 - \gamma_2(\eta) \overline{\nu_{K_2}}) \|p_2\|^2;$$

$$l(p)^T \triangleq \begin{bmatrix} \|D\varphi\| \\ \|p_2\| \end{bmatrix}; Q(p) \triangleq \begin{bmatrix} \overline{\nu_{F_*}} & \frac{1}{2} \mu_{F^T M K_2}(\|p\|) \\ \frac{1}{2} \mu_{F^T M K_2}(\|p\|) & \gamma_0(\eta) \overline{\nu_{K_2}} \end{bmatrix}.$$

The second term is negative as long as γ_2 is greater than the constant function,

$$\gamma_2 > \frac{1}{\overline{\nu_{K_2}}} \quad (13)$$

where $\overline{\nu_{K_2}}$ is specified in Lemma 2. The 2×2 matrix is negative definite when

$$\begin{aligned} \gamma_0 &> 1 + \left(\frac{1}{2\overline{\nu_{F_*}} \overline{\nu_{K_2}}} \right) v_{F^T M K_2}^2 \circ \lambda_\eta^{-1} \\ v_{F^T M K_2}(\chi) &\triangleq \sup_{\|p\| \leq \chi} \sup_{\|p_1\|=1} \|F^T M K_2(p_1, p_2)x\|. \end{aligned} \quad (14)$$

which satisfies the earlier condition (12) as well. ⁶ Under these conditions, $\dot{\theta}$ is negative except at the critical points of φ as identified in the zero section of TJ — the equilibrium states of f .

□

In the next section we consider the application of this construction to disturbed mechanical systems on a configuration space which is guaranteed to admit Morse functions with a single critical point — an open disk in \mathbb{R}^n . The application to non-autonomous mechanical systems on a configuration space with non-unity Euler characteristic — e.g., the punctured manifolds which result from robot obstacle avoidance problems with moving obstacles — is the subject of a separate paper.

4 Application to a PD Controlled Mechanical System

There are relatively few occasions when J admits a Morse function with a single critical point. In this section we examine in detail one such (practically significant) case where J is an open n -disk. In such a case, $\|d\varphi\|$ being radially unbounded, $\dot{\theta}$ is radially unbounded as well.

All symbols will refer to the local coordinate representation. The kinetic energy remains in the general form,

$$\kappa(p) \triangleq p_2^T M(p_1) p_2,$$

where M is analytic in p_1 , and takes values in the set of positive definite symmetric matrices. The only further restriction (imposed for computational convenience, and because it corresponds the situation in robotics) is that both M and its derivatives are bounded on phase space, so that there exists an upper bound, $\overline{\mu_M}$, on the the morphism norm as well as a lower bound, $\overline{\nu_M}$, and there is a linear spread estimate with constant $\sigma_M \triangleq \sqrt{\overline{\mu_M}/\overline{\nu_M}}$ as defined in [4].

The potential energy is defined to be that of a "perfect" Hook's law spring,

$$\varphi(p_1) \triangleq p_1^T K_1 p_1,$$

where K_1 is a positive definite symmetric matrix. The dissipative field is defined to be a Rayleigh field, $k_2(p) p_2 \triangleq K_2 p_2$, where K_2 is a positive definite symmetric matrix as well. There is an external force, as well. Note that the operator bounds on K_1, K_2 are all constant, according to the definitions in [4].

This corresponds to a "linear proportional and derivative feedback compensator", applied to a pure mechanical system (1), the entire closed loop subsequently "forced" by the control input, $u(t)$, giving rise to the nonlinear control system, (9), where K_1, K_2 are constant positive definite symmetric matrices.

4.1 Construction of a Strict Lyapunov Function

Applying the definitions from Section 3.1, the various bounding functions associated with φ are

$$\lambda_\varphi^{-1}(\chi) = \sqrt{\frac{\chi}{\nu_{K_1}}}; \quad v_{\varphi\varphi}(\chi) = \mu_{K_1} \chi; \quad v_{\varphi\varphi}^2 \circ \lambda_\varphi^{-1}(\chi) = \mu_{K_1} \sigma_{K_1}^2 \chi.$$

The construction of θ depends upon the angle map, α , which, in turn, requires the choice of a morphism, $F \in M^\omega[TJ, TJ]$, whose symmetric part is positive definite. In the present case a convenient choice is specified as $F : (p_1, p_2) \mapsto (p_1, K_1^{-1} p_2)$, for this leads to the simple angle map (??)

$$\alpha \triangleq D\varphi F^T M p_2 = p_1^T M p_2.$$

The coefficient functions, γ_i , may all be chosen constant as shown in [4]. The morphism, $B(x)$, is now computed as

$$\begin{aligned} \nabla_x F d\varphi &= (D_{p_1} F D\varphi) x + \Gamma(F D\varphi, x) \\ &= F D^2 \varphi x + \dot{F}(D\varphi) x + \frac{1}{2} M^{-1} (\dot{M} + J) (F D\varphi) x \\ &= I + 0 + \frac{1}{2} M^{-1} (\dot{M} + J) (p_1) x, \end{aligned}$$

and it follows that

$$\mu_{(MB)_*}(\|p_1\|) = \mu_M(\|p_1\|) + \frac{1}{2} \mu_{(\dot{M})}(\|p_1\|).$$

Applying the assumption that M and DM^S have bounded magnitude on the configuration space yields the affine function

$$v_{(MB)_*}(\chi) \triangleq \overline{\mu_M} + \frac{1}{2} \overline{\mu_{(\dot{M})}} \chi \geq \mu_{(MB)_*}(\chi),$$

⁶Note that $\gamma \circ \eta$ is still C^1 with these definitions, following the same argument as in Proposition 1.

hence

$$\int_0^x v_{(MB)_*} \circ \lambda_\eta^{-1}(s) ds \leq \bar{\mu} M \chi + \frac{3 \bar{\mu}(M)}{2 \sqrt{\bar{\nu} M}} \chi^{3/2}$$

The complete Lyapunov function is given by

$$\vartheta \triangleq p^T P p; \quad P \triangleq \begin{bmatrix} (\gamma_0 + \gamma_R \eta^{1/2}) K_1 & \frac{1}{2} M \\ \frac{1}{2} M & (\gamma_0 + \gamma_R \eta^{1/2}) M \end{bmatrix}; \quad (15)$$

where the constant coefficients are defined in [4]. Compare this to the total energy function,

$$\eta \triangleq p^T H p \quad H \triangleq \begin{bmatrix} K_1 & 0 \\ 0 & M \end{bmatrix}.$$

4.2 Input/Output Analysis of the PD Compensated System

In this section we will refer extensively to the language and detailed computation in the technical report [4] in order to extract explicit relationships between input and output bounds from the globally strict Lyapunov function, ϑ .

Although it was shown in Theorem 1 that the construction has a negative definite derivative along the motions of the pd compensated unforced mechanical system (9), we are now interested in displaying an explicit quadratic lower comparison function for the specific case defined by (15).

Lemma 4 *The negative derivative map, $-L_f(\vartheta)$, admits a quadratic lower comparison function,*

$$\lambda_\vartheta(\chi) \triangleq \nu_Q \chi^2,$$

where Q is a positive definite symmetric matrix defined by the "feedback gains", K_1, K_2 , and "energy scaling constants", $\gamma_P, \gamma_P, \gamma_Q$, used in (15).

The lower comparison function, λ_ϑ , may now be used to find a " β -descent region" (defined in [4]) — i.e. a region wherein the Lyapunov function, ϑ , and, ultimately, the distance to the origin, along solutions to the forced system 9 is guaranteed to be decreasing as long as the forcing function is bounded by β .

Proposition 2 *For every $\beta > 0$ there exists a $\delta > 0$ such that if $\|u\| \leq \beta$ then $\dot{\vartheta}_u$ is strictly negative on the set*

$$\mathcal{P} - \mathcal{P}_\delta \triangleq \{p \in \mathcal{P} : \|p\| \geq \delta\}$$

This result leads immediately to the construction of a "descent rate estimate" as defined in [4], which, in turn, demonstrates that ϑ is not only globally strict, with a radially unbounded Lie derivative, but, when restricted to any compact domain, is uniform as well.

Proposition 3 *For all $\chi_1 \in \mathbb{R}^+$, ϑ , as defined in Proposition 1, is a uniform lyapunov function for f on $\mathcal{D}_3 \triangleq \mathcal{P}_0^{\chi_1/\sigma_P}$*

This yields the desired bound on the magnitude of the state of the forced system, expressed in terms of the "feedback gain" magnitudes and scaling coefficients.

Theorem 2 *The motion of a PD compensated mechanical system (9) from the initial condition $\|p_0\| = \chi_0$, in response to inputs of magnitude less than $\alpha < \infty$ is bounded by*

$$\|p(t)\| \leq e^{-\frac{1}{2}\psi_0(\chi_0)t} \sigma_P \chi_0 + \frac{\sigma_0}{\nu_Q} \beta(\alpha).$$

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