Global Robot Task Specification and Control
Via Objective Functions on the Euclidean Group:
End-Point Tasks

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1 An End-Point Objective Function on the Euclidean Group

The workspace of a robot arm is a compact set, \( \mathcal{W} \subseteq SO(3) \times \mathbb{R}^3 \), within the group of rigid transformations or "euclidean group". Thus, the general project of robot arm command via vector fields on workspace confronts from the very start the question of how to construct computationally effective vector fields on the euclidean group. In this section the task of moving to a desired point in workspace is translated into the problem of global "error equations" in \( \mathcal{W} \). Namely, a class vector fields is sought under whose flow any specified "end-point", \( \mathbf{w}_d \in \mathcal{W} \) can be made globally asymptotically stable. Since it is desirable that the error equations lead directly to a feedback structure guaranteed to produce the same limit behavior in the robot control system (??) we restrict our attention to non-degenerate gradient vector fields, as described in Section ??.

It will be seen that purely topological arguments preclude the possibility of constructing any nondegenerate vector field with a single globally asymptotically stable equilibrium state. Moreover, the restriction to nondegenerate gradient vector fields precludes the possibility of a candidate with less than four equilibrium states. In Section ?? it is shown that the some intuitively appealing error functions do not work. In section ?? it is shown that a construction using the quaternion representation of rotations achieves the "best" possible measurement of error. Namely, a quadratic objective function on \( \mathbb{R}^4 \) results in a gradient vector field on the set of rotations with a single isolated local minimum — the arbitrary desired rotation — and three other isolated unstable equilibrium states. As has been discussed above, in consequence of the relative simplicity of gradient dynamics this is enough to insure that all initial conditions disjoint from a set of measure zero converge toward the desired end-point.

It is worth noting again, as in the introduction, that this workspace planning methodology could be employed in the context of alternative control strategies. For instance, the "resolved acceleration" tracking paradigm introduced by Luh, Walker, and Paul [?] prescribes a torque command designed to cancel the intrinsic nonlinear dynamics of (??) and effect a set of globally asymptotically stable error equations. Instead of relying upon error equations which are locally linear and asymptotically stable, it is possible to design the torque to induce the nonlinear "almost" globally asymptotically stable error equations developed here.

1.1 Morse Functions on the Group of Rotations

Unfortunately, it is not possible to construct a vector field on \( SO(3) \) with a globally asymptotically stable equilibrium state. Since the set is compact, every (non-constant) continuous function has distinct maximum and minimum points. Thus any gradient vector field possessed of an attractive equilibrium state will have a repelling equilibrium state as well. This fact remains true even removing the restriction to gradient systems.

**Proposition 1** Every vector field on \( SO(3) \) possessed of a nondegenerate attracting equilibrium state must have at least one other distinct equilibrium state as well.

**Proof:** Since \( SO(3) \) is a compact odd dimensional manifold without boundary, its Euler characteristic is zero [?]. It follows from the Theorem of Hopf that the sum of the vector field indices at the equilibrium states is zero as well [?]. Let \( R_0 \in SO(3) \) be a non-degenerate isolated attracting equilibrium state of the vector field, \( f \). It will suffice to show that the index of \( f \) at \( R_0 \) cannot be zero.
If $f$ is non-degenerate then $R_0$ cannot be an attracting point unless $Df(R_0)$ has eigenvalues with negative real part. Since the index of a vector field at a non-degenerate equilibrium state is given by the sign of the determinant of its jacobian evaluated at that point [7], it follows that the index of $f$ at $R_0$ is $|Df(R_0)| = -1$.

The author is unaware of further obstacles which might preclude the existence of a vector field on $SO(3)$ rendering this proposition exact (i.e. a vector field possessed of exactly two non-degenerate equilibrium states — one an attractor, the other a repellor) and it would be of some interest either to construct one or to demonstrate its impossibility. However, according to the reasoning given in the beginning of this paper, we are interested in the construction of nondegenerate gradient vector fields against which further topological properties conspire to preclude so useful a design. A closed manifold is said to have Lusternik-Schnirelmann category $n$ if it can be covered by no fewer than $n$ closed subsets, each of which being continuously deformable to a single point in the manifold [12]. The relevant result is that a smooth function on a manifold may have no fewer critical points than the Lusternik-Schnirelmann category of that manifold [13,14]. For purposes of this paper, it suffices simply to report that the Lusternik-Schnirelmann category of real projective space, $P^3$, is 4 [11, (p.92)]. This is most significant to the question at hand since the group of rotations may be represented by the latter manifold via quaternions of unit length, $Sp(1) \simeq S^3 \subset \mathbb{R}^4$, with antipodal points identified [15,16,17], i.e.

$$SO(3) \simeq Sp(1)/\{1,-1\} \simeq \{X \subset S^3 : x \in X \iff -x \in X\}.$$ 

There follows the initially discouraging result,

Proposition 2 Every smooth function on $SO(3)$ has at least four distinct critical points.

An appeal to Morse theory as in the previous section revives hope, for if $\{\nu_0,\nu_1,\nu_2,\nu_3\}$ once again denotes the type of a Morse function, there are several combinations of four critical points which satisfy the mandatory relationship,

$$0 = \chi(SO(3)) = \nu_0 - \nu_1 + \nu_2 - \nu_3$$

with $\nu_3 = 1$ — the desired situation corresponding to a single local minimum which will result in a single "almost global" attractor. While there has yet been provided no guarantee, logically speaking, against further topological obstructions, it is now worth attempting to find such a function. To do so in a fashion which yields an effectively computable vector field, it is necessary to work in some coordinate system for $SO(3)$. Three familiar coordinate representations of spatial rotations are the euler angles, the rotation matrices, and the quaternions.

1.1.1 Attempted Construction using Euler Angles and Rotation Matrices

Many robots are built with a "spherical wrist" — a kinematic map which "decouples" position from orientation,

$$W : q \mapsto \begin{bmatrix} R(q_2) & r(q_1) \\ 0^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$
In such cases, the three wrist variables \( \varphi = (\theta_1, \theta_2, \theta_3) \), often determine a kinematic map which is equivalent to the euler angles, e.g. \( R : \mathbb{R}^3 \to SO(3) \) given by

\[
R(\theta_1, \theta_2, \theta_3) \triangleq R_0 \exp \{ J_1 \theta_1 \} \exp \{ J_2 \theta_2 \} \exp \{ J_3 \theta_3 \}
\]

where \( \{ J_1, J_2, J_3 \} \) is a basis of \( so(3) \), e.g. as in Appendix ???. It is well known that such maps are surjective on \( SO(3) \) [??], and the natural course of action would seem to suggest the construction of objectives and gradients in these coordinates considered as a subset of \( \mathbb{R}^3 \). Unfortunately, the euler angles are only locally injective, e.g.,

\[
R(\pi, \pi, 0) = R(0, 0, \pi),
\]

and, since it is a globally defined vector field which is required, considerable additional effort would be needed to force the candidate objective function to attain the same value on distinct points of \( \mathbb{R}^3 \) identified as representing the same rotation. Obviously, if the kinematics does not include a “spherical wrist”, this resort is impossible anyway.

On the other hand, consider the possibility of constructing an exact Morse function using rotation matrices. Given a desired orientation, \( R_d \), an obvious choice for measuring errors would be to take advantage of the natural metric in the embedding space \( ^1 \mathbb{R}^9 \), \( ^{\mathbb{R}^9} \),

\[
\varphi_{\varphi}(r) \triangleq \frac{1}{2} ||r - r_d||^2 = \frac{1}{2} \text{tr} \{ (R - R_d) [R - R_d]^T \}.
\]

It is worth noting that this construction has at least one intuitively appealing property with respect to the embedded rotations. Namely, if \( \theta \) is the angle of rotation (in the plane orthogonal to its fixed axis) of the “error” rotation, \( R_d R_d^T \), then

\[
\varphi_{\varphi}(r) = \text{tr} \{ I - R_d R_d^T \} = 2[1 - \cos(\theta)]
\]

measures the distance of this angle to zero or any multiple of \( 2\pi \).

It is clear that \( R_d \) is a critical point of \( \varphi \), and it is not too hard to show that it is an isolated minimum — a nondegenerate critical point with Morse index 0. Unfortunately, there is a continuum of critical points at the “meridian” of \( SO(3) \) so \( \varphi_{\varphi} \) is useless to the task at hand.

Lemma 3  The squared euclidean distance function restricted to the group of spatial rotations represented as a submanifold of \( \mathbb{R}^9 \), \( \varphi_{\varphi} |_{SO(3)} \), has a continuum of critical points.

Proof:  For any \( R \in SO(3) \), label the columns \( R = [x, y, z] \). The critical points of \( \varphi_{\varphi} |_{SO(3)} \) are exactly the zeroes of its differential \( d \left( \varphi_{\varphi} |_{SO(3)} \right) \in T^*SO(3) \subset T^* \mathbb{R}^9 \), whose coordinate representation at any point, is obtained by projecting \( D_r \varphi_{\varphi} = [r - r_d]^\top \) onto \( T^*SO(3)_R \). According to Lemma ??, in Appendix ??, a representation as a co-vector on \( \mathbb{R}^3 \) using the coordinate system given by the isomorphism \( TSO(3)_R \simeq so(3) \simeq \mathbb{R}^3 \), is

\[
d\varphi_{\varphi} |_{SO(3)} (R) = (J(x)[x - x_d] + J(y)[y - y_d] + J(z)[z - z_d])^\top,\]

\[= -[J(x)x_d + J(y)y_d + J(z)z_d]^\top.
\]

\(^1\)For the remainder of this paper, denote the stack representation of an array by the lower case letter — i.e. \( r \triangleq R^8 \).
Now \( d \left( \varphi_e \big|_{SO(3)} \right) (R) = 0 \) at some point, \( R \in SO(3) \) if and only if its action upon each element of any basis for \( T_{R} SO(3) \cong \mathbb{R}^3 \) is zero. In particular, consider the basis \( \{x, y, z\} \) specified by the columns of \( R \) itself:

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}^T = \begin{bmatrix}
J(x)^T x_d + J(y)^T y_d + J(z)^T z_d \\
J(x)^T x_d + J(y)^T y_d + J(z)^T z_d \\
J(x)^T x_d + J(y)^T y_d + J(z)^T z_d
\end{bmatrix}^T R = \begin{bmatrix}
- z^T y_d + y^T z_d \\
z^T x_d - z^T z_d \\
y^T x_d + x^T y_d
\end{bmatrix}^T.
\]

But the latter is equivalent to the condition that \( R^T R_d \) be a symmetric matrix a condition met by all the "pure quaternion" rotations (i.e. those with zero real part — see Appendix ??) whose representation as \( 3 \times 3 \) rotation matrices may be written \( Q(e) \triangleq 2ee^T - I \in SO(3) \) where \( e \in S^2 \). Thus, the set of critical points of \( \varphi_e \) distinct from \( R_d \) is \( R(e) \triangleq R_d Q(e) \) — a connected submanifold of dimension 2 in \( SO(3) \).

\( \Box \)

The obvious modifications of this error function — generalized metrics of \( \mathbb{R}^3 \) —

\[ \varphi_e \triangleq [r - r_d]^T P[r - r_d], \]

where \( P \triangleq [Q \otimes S]^T \) is a positive definite matrix in \( \mathbb{R}^{9 \times 9} \) constructed from positive definite matrices in \( \mathbb{R}^{3 \times 3} \) give rise to similar problems.

1.1.2 Construction in Quaternions

Following a suggestion of Professor W. S. Massey [?], it is possible to exhibit a "perfect" Morse function on \( SO(3) \) via a construction using quaternions.

Let \( Q \in \mathbb{R}^{4 \times 4} \) be a positive definite symmetric matrix with distinct eigenvalues, and define a scalar valued map on \( \mathbb{R}^4 \)

\[ \varphi_q \triangleq e^T Q e \]

Proposition 4 (Massey [?]) \( \varphi_q \) is a Morse function on \( P^3 \) possessed of exactly four critical points corresponding to the four eigenvectors of \( Q \). The Morse index at each critical point is determined by the signs of the differences between its corresponding eigenvalue and the three other eigenvalues of \( Q \).

Proof: First note that \( \varphi_q(-e) = \varphi_q(e) \), hence \( \varphi_q \) is a well defined scalar map on \( P^3 \). The critical points of \( \varphi_q |_{S^3} \) are exactly the zeroes of its differential

\[ d\varphi_q |_{S^3} \in T^* S^3 \subset T^* \mathbb{R}^4 \]

whose coordinate representation at any point, is obtained by projecting

\[ [D_e \varphi_q](e) = 2 [Q e]^T \]

onto \( T^*_e S^3 = \left( \text{Ker } e^T \right)^* \). But \( [Q e]^T \) fails to have a non-zero component in \( \left( \text{Ker } e^T \right)^* \) if and only if \( e \) is an eigenvector of \( Q \). Since \( Q \) has exactly four distinct (orthogonal) eigenvalues, it follows that \( \varphi_q |_{S^3} \) has exactly four isolated critical points.
It remains to show that these are non-degenerate critical points. Letting

\[ E \triangleq [e_1, e_2, e_3, e_4] \]

denote the array whose columns are the four eigenvectors of \( Q \), it follows that \( Q \) has the matrix representation

\[ Q = E^T \begin{bmatrix} \xi_0 & 0 \\ \vdots & \ddots \\ 0 & \xi_3 \end{bmatrix} E. \]

The map \( g_i : \mathbb{R}^3 \to S^3 \) given by

\[ g_i(q) \triangleq e_i(1 - q^T q)^{1/2} + e_j q_1 + e_k q_2 + e_l q_3 \]

is a local diffeomorphism between a neighborhood of \( 0 \in \mathbb{R}^3 \) and a neighborhood of \( e_i \in S^3 \) since its Jacobian

\[ D_q g = [e_j, e_k, e_l] + \frac{1}{[1 - q^T q]^{1/2}} e_q^T \]

has rank 3 at \( q = 0 \). Thus, it suffices to show that

\[ \tilde{\varphi}_q \triangleq \varphi_q \circ g_i = \xi_i(1 - q^T q)^{1/2} + \xi_j q_1 + \xi_k q_2 + \xi_l q_3 \]

is non-degenerate at \( q = 0 \). Since

\[ [D_q^2 \tilde{\varphi}_q](0) = \begin{bmatrix} \xi_j - \xi_i & 0 \\ \xi_k - \xi_i & \xi_l - \xi_i \end{bmatrix} \]

it follows that \( \varphi_q \) is non-degenerate at \( e \). Note that the Morse index at this critical point may be set to 0, 1, 2, or 3, according to whether \( \xi_i \) is less than all the other eigenvalues, less than all but one other eigenvalue, greater than all but one eigenvalue, greater than all. Similarly, the Morse index at each successive critical point is assigned by choosing the relative magnitudes of the eigenvalues.

\( \square \)

This result may be used to construct "almost globally convergent" error equations around a desired orientation, \( R_d \) as follows. According to the results of Appendix ??, if \( R = \rho(e) \in SO(3) \) is the rotation matrix representation of a unit quaternion then there is an invertible linear map on \( \mathbb{R}^{10} \), \( H \), such that

\[ \begin{bmatrix} R^8 \\ 1 \end{bmatrix} = \begin{bmatrix} h(e) \\ e^T e \end{bmatrix} = H P_+ (e \otimes e) \]

where \( P_+ \) is the canonical projection onto the equivalence classes of symmetric operators in \( \mathbb{R}^{4 \times 4} \),

\( P_+ : \mathbb{R}^{4 \times 4} / \text{Ker} I - T \)
defined in terms of the "transpose" operator, $T$, introduced by Lemma ?? . It follows that all the squared terms of the quaternion corresponding to $R$, $\bar{P}_+ (e \otimes e)$, may be recovered from the inverse,

$$H^{-1} \begin{bmatrix} R^g \\ 1 \end{bmatrix},$$

and, since the desired objective function requires only these squared terms,

$$\varphi_q(e) = (Q^g)^T (e \otimes e)$$

it may be rewritten as an affine function on the matrix representation of $SO(3)$,

$$\varphi_q(e) = (Q^g)^T H^{-1} \begin{bmatrix} r \\ 1 \end{bmatrix} \triangleq \varphi_q(r),$$

on $\mathbb{R}^9$.

1.2 Composition with Robot Kinematics

Now consider the forward robot kinematic map $g : J \rightarrow \mathcal{W}$,
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I am indebted to Professor W. S. Massey for his contributions to this paper cited in the text, as well as his kind availability and patience over the course of several technical discussions.
A The Stack Representation and Kronecker Products

While most of the material in this section has been available for nearly a century, it seems never to have attained wide-spread familiarity within the engineering and applied mathematics community. Readers may consult Bellman [?] for a nicely motivated presentation which leaves many of the proofs to the reader. On the other hand, MacDuffee [?] offers a more thoroughgoing presentation which makes actual reference to the original work of the late nineteenth century mathematicians who developed these results.

If $A \in \mathbb{R}^{n \times m}$, the "stack" representation of $A \in \mathbb{R}^{nm}$ formed by stacking each column below the previous will be denoted $A^S$ [?].

If $B \in \mathbb{R}^{p \times q}$, and $A$ is as above then the kronecker product of $A$ and $B$ is

$$A \otimes B \triangleq \begin{bmatrix}
a_{11}B & \ldots & a_{1m}B \\
a_{21}B & \ldots & a_{2m}B \\
\vdots & \ddots & \vdots \\
a_{n1}B & \ldots & a_{nm}B
\end{bmatrix} \in \mathbb{R}^{np \times mq}.$$

The kronecker product is not, in general, commutative. Note that while the transpose "distributes" over kronecker products [?],

$$(A \otimes B)^T = (A^T \otimes B^T),$$

the stack operator, in general, does not.

Lemma 5 If $A \in \mathbb{R}^{n \times m}$ then there exists a nonsingular linear transformation of $\mathbb{R}^{nm}$, $T$, such that

$$(A^T)^S = TA^S$$

Proof: For $p = nm$, let $B \triangleq \{b_1, \ldots, b_p\}$ denote the canonical basis of $\mathbb{R}^p$ — i.e., $b_i$ is a column of $p$ entries with a single entry, 1, in position $i$, and the other $p - 1$ entries set equal to zero. The transpose operator is a reordering of the canonical basis elements, hence may be represented by the elementary matrix,

$$T \triangleq \begin{bmatrix}b_1, b_{n+1}, b_{2n+1}, \ldots, b_{(m-1)n+1}, b_2, b_{n+2}, b_{2n+2}, \ldots, b_{(m-1)n+2}, \ldots, b_n, b_{2n}, b_{3n}, \ldots, b_{mn}\end{bmatrix}$$

For $n = m$, if we define $P_+ \triangleq I + T$, $P_- \triangleq I - T$ then both operators are projections onto the set of "skew-symmetric", "symmetric" operators of $\mathbb{R}^n$, respectively, since $P_+^2 = P_+$. Note that $\text{Ker } P_\pm = \text{Im } P_\mp$.

The kronecker product does "distribute" over ordinary matrix multiplication in the appropriate fashion.

Lemma 6 ( [? ] ) If $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{m \times k}$, $D \in \mathbb{R}^{q \times l}$ then

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$
Lemma 7 ([?]) If $B \in \mathbb{R}^{m \times p}$, $A \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times q}$ then

$$[ABC]^S = (C^T \otimes A)B^S.$$

Noting that for any column, $c \in \mathbb{R}^{p \times 1}$, we have

$$c^S = [(c^T)^S] = c,$$

there follows the corollary

Corollary 8 If $B \in \mathbb{R}^{m \times p}$, $c \in \mathbb{R}^p$ then

$$Bc^S = (c^T \otimes I)B^S = ([Bc]^T)^S = (c^TB^T)^S = (I \otimes c^T)(B^T)^S.$$

Noting, moreover, that

$$\text{tr} \{A\} = (I^S)^T A^S,$$

there follows the additional result

Corollary 9 If $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times m}$ then

$$\text{tr} \{AB^T\} = (A^S)^T B^S.$$

Proof:

$$\text{tr} \{AB^T\} = (I^S)^T (AB^T)^S = (I^S)^T (B \otimes I)A^S = (A^S)^T (B^T \otimes I)I^S = (A^S)^T B^S.$$

Lemma 10 ([?]) For any two square arrays, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$,

$$\text{spectrum}(A \otimes B) = \text{spectrum}(A) \cdot \text{spectrum}(B),$$

i.e., every eigenvalue of $(A \otimes B)$ is the product of an eigenvalue of $A$ with an eigenvalue of $B$. 
B Some Useful Computation Concerning Matrix Representations of $SO(3)$

We may represent the group of spatial rotations as a subgroup of $3 \times 3$ arrays,

$$SO(3) \simeq \{ R \in \mathbb{R}^{3\times 3} : R^t R = I, \text{ and } |R| = 1 \}.$$ 

Since $SO(3)$ is a Lie Group, its tangent space at the identity is a Lie Algebra,

$$so(3) \overset{\Delta}{=} T_I SO(3),$$

which admits representation as the set of $3 \times 3$ skew-symmetric matrices,

$$so(3) \simeq \{ J \in \mathbb{R}^{3\times 3} : J^T = -J \}.$$ 

This algebra is defined over a linear vector space isomorphic to $\mathbb{R}^3$ for which a useful basis is specified by the three "unit vectors"

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and for any $u \in \mathbb{R}^3$ we will denote its image in $so(3)$ under this isomorphism as

$$J(u) \overset{\Delta}{=} u_1 J_1 + u_2 J_2 + u_3 J_3,$$

where $u_i$ denote the coordinates of $u$ with respect to any basis. Since the tangent space at any point of a Lie Group is a copy of the Lie Algebra under left (or right) translation, matrix representations of the tangent space of $SO(3)$ at some particular rotation, $R$, may be specified by

$$T_R SO(3) = \{ J(R) : J \in so(3) \},$$

and, thus, are isomorphic to $\mathbb{R}^3$.

Using the isomorphism $\mathbb{R}^3 \simeq so(3)$ described above, it is easy to see that for all $u, v \in \mathbb{R}^3$,

$$J(u)v = u \times v,$$

where the right hand side denotes the vector cross product. Moreover, since

$$J(Ru) = RJ(u)R^t,$$

for all $u \in \mathbb{R}^3, R \in SO(3)$,

$$R(u \times v) = RJ(u)v = J(Ru)Rv = Ru \times Rv,$$

as well [?]. The following additional facts follow by direct computation.

**Lemma 11** For all $u, v \in \mathbb{R}^3$,

$$J(u)J(v)^T = u^T v I - vv^T.$$
Lemma 12 Let $A = [a, b, c]$ be the matrix representation of a nonsingular linear operator on $\mathbb{R}^3$. Then

$$A^{-1} = \frac{1}{[a, b, c]} [b \times c, c \times a, a \times b]^T$$

Finally, these considerations afford a convenient representation of the Cotangent vector space of $SO(3)$ at any point, $R$, as well.

Lemma 13 Given a point, $R \in SO(3)$, represented as the matrix with columns $R = [x, y, z] \in \mathbb{R}^{3 \times 3}$, the projection of any co-vector, $H \in T^*_R \mathbb{R}^{3 \times 3}$ represented as the matrix with columns $H = [k, l, m] \in \mathbb{R}^{3 \times 3}$, onto the subspace $T^*_R SO(3) \subset T^*_R \mathbb{R}^{3 \times 3}$, has the representation

$$[J(x)k + J(y)l + J(z)m]^T$$

with respect to the coordinate system obtained by the isomorphisms $TSO(3)_R \simeq so(3) \simeq \mathbb{E}^3$.

Proof: If $x \in TSO(3)_R$, then its coordinate representation in $T^R \mathbb{R}^3$ is given as $(J(u)R)^T = (I \otimes J(u))R$. Thus the action of any co-vector $h^T = (k^T, l^T, m^T) \in T^* \mathbb{R}^3$ upon the vector $v \in TSO(3)_R \subset T^R \mathbb{R}^3$ may be computed as

$$tr (k^T \mathbb{I}J^T) = tr \left( k^T \mathbb{I} (I \otimes J(u)) \right) = k^T J(u)x + l^T J(u)y + m^T J(u)z = [J(x)k + J(y)l + J(z)m]^T u.$$

C Quaternion Representation of $SO(3)$ and Related Algebra

The quaternions $^2$ are the four-tuples, $\{ e \in \mathbb{R}^4 \}$, comprised of a real part, $\epsilon \in \mathbb{R}^1$, and a vector part, $e \in \mathbb{R}^3$, endowed with a binary product

$$e_1 e_2 \triangleq \begin{bmatrix} \epsilon_1 \epsilon_2 - e_1^T e_2 \\ \epsilon_1 e_2 + e_2 e_1 + e_1 \times e_2 \end{bmatrix}$$

which is associative (but not commutative) and distributive over addition; possesses a "unit", $(1, 0, 0, 0)^T$, and a "zero" $(0, 0, 0, 0)^T$; and preserves euclidean norm, $\| e \| \triangleq \sqrt{e^T e} \text{ -- i.e.,}$

$$\| e_1 e_2 \| = \| e_1 \| \| e_2 \|.$$  

$^2$This paragraph is adapted from conversations with and unpublished notes of Professor W. S. Massey $^3$. Useful references for this material include $^?, ^?, ^?$. 

$^3$
Defining the *conjugate* of a quaternion,

\[ \bar{e} \triangleq \begin{bmatrix} e \\ -e \end{bmatrix}, \]

we have

\[ e\bar{e} = \begin{bmatrix} |e|^2 \\ 0 \end{bmatrix}, \]

and hence, every non-zero quaternion, \( e \neq 0 \), has an "inverse",

\[ e^{-1} \triangleq \frac{1}{|e|^2}. \]

One customarily calls the three dimensional subspace \( \text{Ker}[1, 0, 0, 0] \), the set of *pure quaternions*, while its orthogonal complement consists of the *scalar quaternions*.

Given any non-zero quaternion, \( e \neq 0 \) the "inner automorphism"

\[ \xi_e : f \mapsto efe^{-1} \]

is a bijective linear (in \( f \)) transformation of \( \mathbb{R}^4 \) which preserves addition, multiplication, and norm. Thus, \( \xi_e \in O(4) \), and it follows that the unit sphere, \( S^3 \subset \mathbb{R}^4 \) is invariant under all inner automorphisms. Two quaternions define the identical inner automorphism, \( \xi_{e'} = \xi_e \), if and only if they lie on the same line of \( \mathbb{R}^4 \), \( e' = \alpha e \), for some \( \alpha \in \mathbb{R}^4 \setminus \{0\} \). Thus, the set of inner automorphisms on the quaternions may be identified with projective three space,

\[ \{ \xi_e : e \neq 0 \} \cong \mathbb{P}^3 \triangleq \left\{ \mathcal{X} \subset S^3 : x \in \mathcal{X} \iff -x \in \mathcal{X} \right\}. \]

Simple computation reveals that the pure quaternions constitute an invariant subspace of every inner automorphisms, \( \xi_e \). Since the set of pure quaternions is isomorphic to \( \mathbb{R}^3 \) and \( \xi_e \) preserves euclidean norm as mentioned above, it now follows that the restriction has the property

\[ \xi_e |_{\text{Ker}[1,0,0,0]} \subset O(3) \]

of an orthogonal transformation of \( \mathbb{R}^3 \). With some more effort, it can be shown that \( \mathcal{R} \) is exactly a copy of \( SO(3) \) in the sense of the following result.

**Proposition 14** (\( [7,7,7] \)) *There exists a group isomorphism, \( \rho \), between \( SO(3) \) and the quotient group of unit norm quaternions with antipodal points identified.*

However, rather than reviewing the standard mathematical presentation of this result, it will be more useful here to develop a computational version. Namely, we seek a map between \( 4 \times 1 \) arrays and \( 3 \times 3 \) arrays, \( h : \mathbb{R}^4 \rightarrow \mathbb{R}^{3\times3} \), whose restriction to the equivalence classes of antipodal unit quaternions in the former space, \( \rho \triangleq h|_{\mathbb{P}^3} \), takes its image in the rotation matrices of the latter space, is a homomorphism between the two, and admits computable inverse on the restricted image set.

Define the map \( [?], \)

\[ h(e) \triangleq ee^T + [eI - J(e)]^2. \]
Note that since this is quadratic in $e$, the image of antipodal points of $S^8$ is indeed identical as required if $\rho(x)$ is to be well-defined on its domain, $P^3$. It may be verified by computation that $[h(e)]^T h(e) = (e^T e)^2 I$, and $[h(e)] = (e^T e)^3 = 1$, hence, $h|_{S^8}$ takes its image in the matrix representation of $SO(3)$ . In fact, the map is injective on $P^3$ as may be further verified by computation. Finally, computation verifies as well that this is a homomorphism — i.e.

$$\rho(x_1 x_2) = \rho(x_1) \rho(x_2).$$

We may now construct the inverse function.

**Lemma 15**

$$h(e)^8 = H (ee^T)^8,$$

where $H$ is a surjective linear operator.

**Proof:** Notice that

$$h(e) = ee^T + e^2 - J(e) j(e)^T e - 2c j(e)$$

$$= 2Nee^T N^T + I_{3 \times 3} e^T Le - 2c j(e)$$

$$= 2Nee^T N^T + I_{3 \times 3} (L^8)^T (ee^T)^8 - 2c j(e),$$

where

$$N \triangleq [0, I_3]; \quad L \triangleq \begin{bmatrix} 1 & 0^T \\ 0 & -I_3 \end{bmatrix},$$

so that

$$h(e)^8 = 2(N \otimes N) (ee^T)^8 + (I_{3 \times 3})^8 (L^8)^T (ee^T)^8 - 2J (ee^T)^8$$

$$= [2(N \otimes N) + (I_{3 \times 3})^8 (L^8)^T - 2J] P_+ (ee^T)^8,$$

$$\triangleq H (ee^T)^8,$$

where $P_+ \triangleq [I_{4 \times 4} + T_{4 \times 4}]$ is the projection into the space of symmetric operators on $\mathbb{R}^4$ defined in consequence of Lemma ?? . Let $S \triangleq 2(N \otimes N) + (I_{3 \times 3})^8 (L^8)^T$. Since $S|_{\text{Im } P_+}$ takes its image in the subspace of symmetric matrices, while $\tilde{J}|_{\text{Im } P_+}$ takes its image in the complementary subspace of skew-symmetric matrices in $\mathbb{R}^{3 \times 3}$, we have $\dim \text{ Im } H = \dim \text{ Im } \tilde{S}|_{\text{Im } P_+} + \dim \tilde{J}|_{\text{Im } P_+} = 6 + 3 = 9$, as may be easily shown by direct computation.

□

**Lemma 16** The linear map $\tilde{H} : \mathbb{R}^{16} \to \mathbb{R}^{10}$ defined by

$$\tilde{H} \triangleq \begin{bmatrix} H \\ (P_{4 \times 4}^8)^T P_+ \end{bmatrix}$$

is surjective.
First note that projective three space may be represented by a subset of non-negative symmetric arrays in $\mathbb{R}^{4 \times 4}$. Letting $\mathcal{P} \triangleq \{ E \in \mathbb{R}^{4 \times 4} : E = E^T \text{ and } e_{ij} \geq 0, i, j = 1, 4 \}$, note that $\eta : \mathbb{P}^3 \rightarrow \mathcal{P}$ defined by

$$\eta : e \mapsto ee^T$$

is injective.

It is often useful to refer to the following basis of $\mathbb{R}^{2 \times 2}$:

$$I \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; J \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; K \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; L \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For example, this provides a simple mechanism for generating coordinate free expressions of orthonormal bases in $\mathbb{R}^4$.

**Lemma 17** Given any $e \in S^3$ the set

$$\mathcal{B}(e) = \{e, e_f, e_k, e_l\} \triangleq \{e, (J \otimes I)e, (K \otimes J)e, (L \otimes J)e\}$$

is an orthonormal basis of $\mathbb{R}^4$.

**Proof:** Since $[J \otimes I]^T = -(J \otimes I); [K \otimes J]^T = -(K \otimes J); [L \otimes J]^T = -(L \otimes J)$ are all skew-symmetric matrices, it follows that $e_f, e_k, e_l$ are each orthogonal to $e$. Moreover, as an example, $[L \otimes J]^T(L \otimes J) = L^T L \otimes J^T J = I \otimes I$, hence $e_f^T e_l = e^T e = 1$ and $e_f, e_k \in S^3$ as well, as may be seen by the analogous computation. Finally, as an example, $e_f^T e_l = e^T [J \otimes I]^T (L \otimes J)e = e^T (J^T L \otimes J)e = e^T (K \otimes J)e = 0$, and all of $e_f, e_k, e_l$ may be seen to be mutually orthogonal by analogous computation.

$\square$