From Stable to Chaotic Juggling:
Theory, Simulation, and Experiments

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M. Bühler and D. E. Koditschek
Center for Systems Science
Yale University, Department of Electrical Engineering

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Abstract

Robotic tasks involving intermittent robot-environment interactions give rise to return maps defining discrete dynamical systems that are, in general, strongly nonlinear. In our work on robotic juggling, we encounter return maps for which a global stability analysis has heretofore proven intractable. At the same time, local linear analysis has proven inadequate for any practical purposes.

In this paper we appeal to recent results of dynamical systems theory to derive strong predictions concerning the global properties of a simplified model of our planar juggling robot. In particular, we find that the local (linearized) stability properties determine the essential global (nonlinear) stability properties, and that successive increments in the controller gain settings give rise to a cascade of stable period doubling bifurcations that comprise a "universal route to chaos". The theoretical predictions are first verified via simulation and subsequently corroborated by experimental data from the juggling robot.

1 Introduction

We have built a one degree of freedom robot that juggles two degree of freedom bodies — pucks falling (otherwise) freely on a frictionless plane inclined into the earth's gravitational field [5]. We have developed the rudiments of a "geometric language" — a family of "mirror laws" that map puck states into desired robot states — capable of translating certain abstract goals (e.g. batting, juggling, catching) into robot control laws whose closed loop behavior has been shown experimentally to accomplish these tasks in a stable and robust manner [2, 7]. We have proven, as well, that these mirror laws are correct by resort to a local stability analysis of linearized closed loop models [3, 4]. Unfortunately, this local analysis and its intrinsically weak conclusions have been of little use in predicting the physical consequences of different gain settings [4]. In this paper, we appeal to recent results of dynamical systems theory to derive strong predictions concerning the global properties of a simplified model of our planar juggling robot, and report a gratifying correspondence between theory and experiment.

We have argued [4] that, as in any other science, experimentation is mandatory in robotics. Blind tinkering with hardware may be short-sighted, but theory detached from reality is futile. Thus, our larger program of research had reached a critical juncture: determining the global stability properties caused by our controllers seemed to be of utmost practical importance; at the same time this determination (in contrast to the straightforward linear analysis) seemed very far away. This paper marshals an array of theoretical tools that gives considerable promise of narrowing the gap between our analysis and practice. We capitalize on a "by-product" of the recently burgeoning study of bifurcations, chaos, and sensitive dependence in qualitative dynamical systems theory [12, 28, 9, 11, 10]. Specifically, we find that certain (seemingly restrictive) sufficient conditions for "reading off" from the derivative at a fixed point the essential global stability properties of an entire dynamical system are met by a simplified model of our closed loop system. Moreover, the coincidence of our systems' stability mechanisms with these special cases may be shared by the underlying stability mechanism of Raibert's hoppers [25, 17]. This
coincidence, if physically intrinsic, holds great promise for advancing the science of robotics in intermittent dynamical environments.

1.1 Allied Literature

For ease of exposition it seems preferable to put the technical terminology used above — bifurcations, intermittent dynamical environments — in the context of its background literature before proceeding to a more detailed synopsis of our present contributions.

1.1.1 Unimodal Return Maps: Universal and Global Properties

A unimodal map is a scalar function with a “single hump.” Such functions arise naturally in biology and many other disciplines [21] where their successive iteration models various evolving systems. Unimodal maps came under increasing scrutiny at the end of the last decade. It was observed numerically and subsequently proven theoretically [11, 9] that dynamical systems defined by many families of such functions exhibit a regular pattern of bifurcations — qualitative changes in steady state properties — as a parameter is increased. These successive bifurcations correspond to ever more complex stable periodic behavior until, eventually, steady state behavior becomes chaotic. This global bifurcation diagram is universal in the sense that genera families exhibit the same diagram with doublings in periodicity at parameter intervals that are asymptotically identical, regardless of the details of the functions!

Amidst the general press of these great theoretical advances can be found two singular contributions that we feel have particular importance for engineers in general and roboticists in particular. The addition of a much more restrictive technical condition — a negative “Schwarzian derivative” — was introduced by Singer [28] and shown to preclude the possibility of more than one stable steady state solution. Subsequent work by Guckenheimer [12] demonstrated that the same technical condition guaranteed an essentially global domain of attraction for this distinguished steady state solution. The great importance of these results is that they relieve us of the necessity of performing any tests concerning the physical extent of stability beyond the straightforward computation of derivatives and their magnitudes.

In contrast, for general nonlinear dynamical systems, the characterization of the domain of attraction of a stable steady state is essentially impossible (from the analytical point of view). The procedure of Zubov [13, §34] may be used to obtain numerical estimates in case a Lyapunov function has been found for a continuous stable dynamical system. We do not know even of an analogous numerical procedure for discrete dynamical systems. Our work in the realm of purely geometric robot tasks [20, 26, 27] makes use of a very particular device — a global version of Lord Kelvin’s century old result concerning the dissipation of total energy [19, 18] — to read the essential global convergence behavior off of local tests. We are greatly encouraged to hope that the Singer-Guckenheimer theory may prove to offer the foundation of an analogous device for the robot tasks addressed here.

Although we are primarily interested in the domain of attraction of stable period one orbits,
the universality of transition to higher period orbits and even chaos holds great interest. In the present paper we use the strong predictions that this theory makes merely to help corroborate the physical validity of our formal reasoning. However, our ability to exhibit experimentally the theoretically predicted higher period fixed points raises the hope that this analysis may eventually be used conversely to tune gains more precisely in practice. In the longer term, one might even imagine robotic tasks that explicitly require higher period orbits or even chaos.

1.1.2 Intermittent Dynamical Environments

The task domain of intermittent dynamical environments — the command and control of robots interacting with a world possessed of independent dynamics as well as geometry [6] — is finally beginning to attract the attention it deserves. The first systematic work in this domain has been the pioneering research of Raibert whose careful experimental studies verify the correctness of his elegant control strategies for legged locomotion [25]. Our analysis of simplified models of Raibert's hopper [7] uses the same global tools as in the simplified model of our juggler to make strong assertions about the transient and limit behavior of his machines (that remain to be empirically validated). On the other hand, McGeer has successfully used the kind of local linearized analysis that failed in our experiments to build passive (unpowered) walking robots [24, 23], and feels that similarly tractable analysis should suffice for controlling running machines as well [22]. Wang [30] has proposed to use the same local techniques for studying open loop robot control strategies in intermittent dynamical environments although his ideas remain to be tested as well. Research by Atkeson et al. on juggling [1] suggests that task level learning methods may relieve dynamics based (or any other parametric) controller synthesis methods of the need to achieve precise performance requirements as long as a basically functioning system has been assured. Thus, increasing numbers of researchers have begun to explore the problems of robotics in intermittent dynamical environments with increasingly successful results.

1.2 Overview of our Present Results

We now sketch the contributions of the present paper and provide an organizational guide to reading about them.

1.2.1 Contributions of This Paper

The chief theoretical contribution of this paper is the formal demonstration that the local stability of a period one orbit in a simplified one degree of freedom model of our closed loop robot-environment dynamical system implies its essential global asymptotic stability. After the appropriate interpretation of the Singer-Guckenheimer theory, this demonstration reduces merely to the presentation of a linear change of coordinates and a computation of the Schwarzi derivative. Thus, we present in Theorem 3 a range of controller gains over which the juggler will always accomplish its task. Furthermore, we show that these gains parametrize the simple model in accordance with the conditions that assure the universal cascade of stable period doubling
bifurcations to chaos. Thus, we predict a region of controller gains beyond that producing stable period one behavior that must give stable period two, then stable period four orbits, and so on. Since each of these stable orbits is guaranteed to be essentially global asymptotically stable, we predict that this steady state behavior must be observed from any initial conditions.

Simulations of the simplified model bear out all the theoretical predictions. Indeed, we produce a bifurcation diagram via computer iterations that might just as easily have been lifted out of a textbook. When we constrain the puck to move in the vertical direction only, then physical experiments bear a striking resemblance to the simulated results, and thus, corroborate the theory almost exactly.

Strictly speaking, the theory applies only to scalar maps, thus cannot immediately generalize to higher than one degree of freedom physical systems. However, we provide crude analytical evidence that the two degree of freedom nonlinear model is “close” to a cross product of the unimodal map in the vertical direction with an attracting map in the horizontal direction. Thus, higher dimensional analogues of the unimodal theory [9] should be applicable. A stable fixed point in the two degree of freedom model exactly encodes the “vertical one juggle” task [3, 2, 4]. When the two degree of freedom model is simulated from initial conditions placed on or near the fixed point in the horizontal direction and arbitrarily far from the fixed point in the vertical direction we obtain a very similar bifurcation diagram. Moreover, analogous physical experiments with the unconstrained puck in the two degree of freedom juggling plane exhibit the same properties! Thus we feel justified in hoping that a little more analysis will result in an unrestricted physically corroborated theory of how to tune the controller gains to achieve the vertical one juggle.

1.2.2 Organization of the Paper

In Section 2 we derive the model of the simple one degree of freedom juggler and present without derivation the generalization to the two degree of freedom juggler. Section 3 summarizes the Singer-Guckenheimer theory and then demonstrates its relevance to our application. Simulations of both the one and the two degree of freedom case verify the predictions of the theory. Finally in Section 4 we present experimental data from both the one and the two degree of freedom system which coincide with the theoretical predictions and the simulations of the previous section.
2 Models

This section, largely devoted to an illustrative one degree of freedom case, portrays more simply than can the two degree of freedom experimental system the modeling process as well as the underlying ideas in our new feedback control law, or “mirror algorithm.” For brevity we will refer to this simple hypothetical juggler in the sequel as the “Gedankenrobot”, although, in point of fact we have actually “instantiated” it by physical modifications to the planar juggler reported in Section 4.1. In Section 2.1 we derive the model for a one degree of freedom juggler. The specification of a feedback control law will then give rise to a scalar return map of puck impact velocities that we analyze in Section 3.2. Both the modeling and the control law generalize in a straightforward fashion to the two degree of freedom juggler. We will merely sketch this generalization as well as the formal statement of what we call the environmental control problem and the robot control problem in Section 2.2. For a detailed discussion of these issues and the complete derivation of the planar juggler model refer to [4, 3, 7].

2.1 A Prismatic Robot Meets a One Degree of Freedom Puck

The simple one degree of freedom juggler is displayed in Figure 1. A puck falls freely in the gravitational field toward a prismatic robot actuator. The robot’s and puck’s positions are denoted by \( r \) and \( b \), respectively. The task — the vertical one-juggle — is to force the puck into a stable periodic trajectory with specified apex point. Since the robot can only provide intermittent impacts to the environment to be controlled — the puck — it makes sense to examine the discrete map between puck states at those interactions as a function of the robot’s inputs. For now, we will ignore the robot’s dynamics and assume it capable of applying arbitrary inputs to the puck during these recurring interactions. We can now examine the “environmental control problem”. Namely, given a sequence of desired puck states comprising the “task,” find a sequence of robot control inputs to achieve that task.

![Figure 1: The Gedankenrobot](image)

2.1.1 The Model with No Friction

First we construct the discrete model that relates two successive puck states \( w = (b, \dot{b}) \) just before impact as a function of the robot control inputs. This process consists of modeling the puck-robot impacts and the puck’s flight dynamics.

For the impact model we make the common assumption that the elastic impact can be
modeled accurately by a coefficient of restitution law [29] and that the robot's velocity \( \dot{r} \) during impact remains unchanged. This is a realistic assumption if the robot's mass is large compared to the mass of the puck, or if the robot is powerful enough to reject the impact disturbances. Then the puck velocity just after impact \( \dot{b}' \) is related to the velocity just before impact \( \dot{b} \) via

\[
b' = -\alpha \dot{b} + (1 + \alpha) \dot{r} = c(\dot{b}, \dot{r}),
\]

where \( \alpha \in (0, 1) \) denotes the coefficient of restitution. We have verified the validity of this simple law empirically on the planar juggler.

For ease of exposition, we will neglect friction during flight in the following derivation and account for its effects in Section 2.1.3. In isolation, the puck's dynamics without friction are given by

\[
\dot{b} = -\gamma,
\]

where \( \gamma \) denotes the gravitational constant. The puck's flight model is obtained by integrating (2) starting with the puck states just after impact, \( \omega' = (b', \dot{b}') \),

\[
\begin{bmatrix}
  b(t) \\
  \dot{b}(t)
\end{bmatrix}
=
\begin{bmatrix}
  b' + \dot{b}' t - \frac{1}{2} \gamma t^2 \\
  \dot{b}' - \gamma t
\end{bmatrix}.
\]

As the impacts are modeled to be instantaneous, the puck position during an impact remains unchanged, \( \dot{b}' = \dot{b} \). If we now combine the impact model (1) with the flight model (3) and select the time of flight and the robot velocity at impact as our robot control inputs, we obtain the discrete map between successive puck impacts as a function of the two robot control inputs,

\[
f(b, \dot{b}, u_1, u_2) = \begin{bmatrix}
  b + c(b, u_2)u_1 - \frac{1}{2} \gamma u_1^2 \\
  c(b, u_2) - \gamma u_1
\end{bmatrix}.
\]

Notice that choosing the time of flight as a robot control input is equivalent to choosing an impact position.

2.1.2 The Mirror Law

The vertical one-juggle task can now be specified as a sequence of desired puck states just before impact. Selecting \( \omega_* = (b^*, \dot{b}^*) \) as the desired constant set point of (4) indicates that the impact should always occur at the position \( b^* \) with the velocity just before impact \( \dot{b}^* \). If \( \omega_* \) is truly a fixed point of the closed loop dynamics, then the velocity just after impact must be \( -\dot{b}^* \), and this "escape velocity" leads to a free flight puck trajectory whose apex occurs at the height \( b_{apex} = b^* + \frac{\dot{b}^*}{2\gamma} \), assuming the simple ballistic model of free flight with no friction (2). Thus, a constant \( \omega_* \) "encodes" a periodic puck trajectory which passes forever through a specified apex point, \( b_{apex} \).

Successful control of the vertical one-juggle task is achieved via a new class of feedback algorithms termed "mirror algorithms" [6]. Suppose the robot tracks exactly the "distorted mirror" trajectory of the puck,

\[
\tau = -\kappa_{10} \dot{b},
\]
where $\kappa_{10}$ is a constant. In this case, impacts between the two do occur only when $(r, b) = (0, 0)$ with robot velocity
\[ \dot{r} = -\kappa_{10} \dot{b}. \]  
(5)

For simplicity we will assume that the desired impact position is always selected to be $b^* = 0$. Notice that any other impact position can be achieved by shifting the coordinate frame for robot and puck to that position. Now solving the fixed point condition $b' = c(b^*, \dot{r}(b^*)) = -\dot{b}^*$ for $\kappa_{10}$ using (1) and (5) yields a choice of that constant,
\[ \kappa_{10} = \frac{1 - \alpha}{1 + \alpha}, \]
which ensures a return of the puck to the original height. Thus a properly tuned “distortion constant,” $\kappa_{10}$ will maintain a correct puck trajectory in its proper periodic course.

Being able to maintain the vertical one-juggle at fixed point condition with such a simple mirror control law is an encouraging first step, but still impractical, as it is not stable. The second idea at work which will assure stability is actually borrowed from Marc Raibert [25], who also uses the total energy for controlling hopping robots. In the absence of friction, the desired steady state periodic puck trajectory is completely determined by its total vertical energy,
\[ \eta(w) = \frac{1}{2} b^2 + \gamma b, \]
in this case,
\[ \eta^* \triangleq \eta(w^*) = \frac{1}{2} b^*^2. \]

This suggests the addition to the the original mirror trajectory,
\[ r = -\kappa_1(w) b; \quad \kappa_1(w) \triangleq \kappa_{10} + \kappa_1[\eta^* - \eta(w)], \]  
(6)
of a term which “servos” around the desired steady state energy level. Thus, implementing a mirror algorithm is an exercise in robot trajectory tracking wherein the reference trajectory is a function of the puck’s state.

At steady state, $\eta(w) = \eta^*$, the fixed point condition is still preserved. However, deviations of $\eta$ away from $\eta^*$ cause proportionately harder or softer robot impacts than the steady state condition requires. It is plausible that these proportionally adjusted deviations will cause convergence toward $\eta^*$: we will prove in Section 3.2 that this is indeed the case.

Assume the desired task point is $w^* = (b^*, \dot{b}^*) = (0, \dot{b}^*)$ — without loss of generality we assume $b^* = 0$. Under mirror law control all impacts must occur on $b^* = 0$. The effective robot control inputs at impact are
\[ u_1 = \frac{2}{\gamma} c(b), \]
\[ u_2 = \dot{r} = -\kappa_1(\dot{b}) \dot{b} - \kappa_1(\dot{b}) b = -\kappa_1(\dot{b}) \dot{b}. \]

Notice that in the $u_2$ equation the last term $\kappa_1(\dot{b}) b$ is zero, because the impact position $\dot{b}$ is always zero, and, in this frictionless case, also $\kappa_1$ vanishes as $\eta \equiv 0$. 

\[ \Gamma = \nabla (\omega) + \rho_{\omega} \]
Substituting these robot control inputs in (4) we obtain the scalar map of puck impact velocities just before impact at the invariant impact position \( b^* = 0 \),

\[
f(\hat{b}) = \hat{b} \left( 1 - \beta (\hat{b}^2 - \hat{b}^{*2}) \right),
\]

where, for ease of exposition, we have defined

\[
\beta = \frac{1 + \frac{\alpha}{\kappa_{11}}}{2}.
\]

2.1.3 Juggling with Friction

In an effort to synthesize a more realistic closed loop map than (7) that will serve us in predicting experimental results we now include coulomb friction between puck and sliding surface. Furthermore, in order to prevent the puck from falling off the sliding plane, we incline the juggler in the gravitational field away from the vertical by an angle \( \delta \), which has the effect of decreasing gravitational acceleration. Now the dynamics of the puck in isolation are described by

\[
\hat{b} = -\gamma \cos \delta - \text{sgn}(\hat{b}) \mu_{fric} \gamma \sin \delta.
\]

Here \( \mu_{fric} \) denotes the friction coefficient for dry friction which is determined empirically as \( \mu_{fric} = 0.16 \). We integrate this equation starting with initial conditions \( w' = (b', \dot{b}') \), the puck states just after impact. We then replace again \( b' = b \) and compute \( \dot{b}' \) from the impact dynamics (1) as before to obtain the new discrete impact map with friction

\[
f(b, \dot{b}, u_1, u_2) = \begin{bmatrix}
b + \frac{1}{2} \frac{c(b, u_2)^2}{\gamma_{up}} - \frac{1}{2} \gamma_{dn} (u_1 - \frac{c(b, u_2)}{\gamma_{up}})^2 \\
-\gamma_{dn} (u_1 - \frac{c(b, u_2)}{\gamma_{up}})
\end{bmatrix}.
\]

Here we defined

\[
\gamma_{up} = \gamma \cos \delta + \mu_{fric} \gamma \sin \delta
\]

and

\[
\gamma_{dn} = \gamma \cos \delta - \mu_{fric} \gamma \sin \delta
\]

which denote the (sign inverted) sum of the gravitational and friction forces during the upward and downward motion, respectively. The robot control inputs are again the linear robot impact velocity \( u_2 \), and the time of flight, \( u_1 \). Note however, that (9) for simplicity assumes that the next hit will occur after the puck has traversed its apex, and thus \( u_1 > \frac{c(b, u_2)}{\gamma_{up}} \), the flight time at the apex.

For a fixed point of the closed loop dynamics \( w^* \), the velocity just after impact must be \( -\hat{b}^* \), and this "escape velocity" leads to a free flight puck trajectory whose apex occurs at the height \( b_{apex} = b^* + \frac{\hat{b}^{*2}}{2\gamma_{up}} \), assuming the more realistic ballistic model of free flight with friction (8).

Proceeding now analogously to Section 2.1.1, we apply the same mirror law (6) and obtain the closed loop impact map corresponding to (7),

\[
f(\hat{b}) = \zeta \hat{b} \left( 1 - \beta (\hat{b}^2 - \hat{b}^{*2}) \right),
\]
where again
\[ \beta = \frac{1 + \alpha}{2} \kappa_{11} \]
and we have defined
\[ \zeta \triangleq \sqrt{\frac{\gamma_{dn}}{\gamma_{up}}} = \sqrt{\frac{1 - \mu_{frict} \tan \delta}{1 + \mu_{frict} \tan \delta}} < 1. \]

Friction shifts the fixed point toward zero,
\[ \dot{b}_{fp} = -\sqrt{b^* + \frac{1 - 1/\zeta}{\beta}} \] (11)
and improves the local stability by decreasing the derivative evaluated at the fixed point,
\[ f'(\dot{b}_{fp}) = 3 - 2\zeta(1 + \beta \dot{b}^*). \] (12)

Here a remark on the domain of \( \dot{b} \) is in place. The impact maps (4) and (9) are only defined for positive velocities after impact \( c(b, u_2) > 0 \). This restricts the vertical puck velocity just before impact \( \dot{b} \) for both cases without and with friction to
\[ \dot{b} \in W \triangleq \left( -\sqrt{b^* + \frac{1}{\beta}}, 0 \right) = (\ddot{b}, 0) \] (13)
or equivalently,
\[ \eta \in \left( 0, \eta^* + \frac{1}{2\beta} \right). \]
Successful juggles can only occur within this proper subset of the real line.

Notice that the contact equation \( r = b = -\kappa_1 b \) has — besides the desired solution \( (r, b) = (0, 0) \) a solution \( r = b \) for \( \kappa_1(w) = -1 \). Because \( \kappa_1 = 0 \) in the frictionless case, this would mean that the robot stays in contact with the puck during the entire flight. This occurs for the impact velocity
\[ \dot{b}_{\text{touch}} = -\sqrt{b^* + \frac{2}{\beta}}, \]
or, equivalently, in terms of total energy
\[ \eta_{\text{touch}} = \eta^* + \frac{1}{\beta}. \]

However, we can ignore this case as \( \dot{b}_{\text{touch}} \notin W \).

2.2 A Revolute Robot Meets a Two Degree of Freedom Puck

The physical planar juggling apparatus consists of a puck, which slides on an inclined plane and is batted successively by a simple revolute "robot" — a bar with billiard cushion rotating in the
The derivation of the closed loop dynamic model can be found in [6, 4, 7] and is omitted here for brevity. The discrete dynamical control system

\[ w_{j+1} = f(w_j, u_j), \]

corresponding to (4) is

\[ f(w, u) = \begin{bmatrix} b + c(b, u_2)u_1 - \frac{1}{2} au^2_1 \\ c(b, u_2) - au_1 \end{bmatrix}. \]  

(14)

Here \( w = [\dot{b}, \ddot{b}]^T = [b_1, b_2, \dot{b}_1, \dot{b}_2]^T, \) \( u = [u_1, u_2]^T, \) \( a = [0, -\gamma \sin \delta]^T, \) and \( \delta \) denotes again inclination of the sliding plane away from vertical. For clarification, these quantities are shown in Figure 3.

The basic idea of the mirror algorithm from the previous section carries over to this environment by adding linear PD feedback compensation terms for the horizontal component. Next, define the "puck angle" as

\[ \theta_p(b) \triangleq \arctan \frac{b_2}{b_1}. \]

Now, as opposed to controlling the robot height as a function of puck height, we control the robot angle \( \theta_r \) as a function of puck angle \( \theta_p, \)

\[
\theta_r(t) = -\kappa_1(w)\theta_p + \kappa_2(w)
\]

\[
\kappa_1(w, w^*) \triangleq \kappa_{10} + \kappa_{11}[\eta(w^*) - \eta(w)]
\]

\[
\kappa_2(w, w^*) \triangleq \kappa_{21}(b_1 - b_1^*) + \kappa_{22}(\dot{b}_1 - \dot{b}_1^*)
\]

(15)

where \( \kappa_{ij} \) are again fixed scalar gains.

For the same reason as the Gedankenrobot impact position is fixed to \( b^* = 0 \) under mirror law control, we find that now, when the robot tracks the mirror trajectory, \((\theta_r, \dot{\theta}_r),\) all impacts must occur on a three dimensional invariant submanifold of the four dimensional puck impact phase space. However, we cannot express this invariant impact submanifold in an explicit representation. Therefore, the impact map corresponding to (10) is not available explicitly in three dimensions, but rather implicitly in four dimensions. A full nonlinear analysis of this system promises to be very difficult. As a first step, we offer a local linear stability analysis on the invariant submanifold around \( w^* \) in [6].
Figure 2: The Planar Juggler

Figure 3: The Impact Event
3 The Stability Properties of Unimodal Maps

We now review the Singer-Guckenheimer theory and present the formal demonstration of its relevance to the present application. Bifurcation plots are generated on the computer as an illustration of the theoretical statements and for purposes of comparison with the experimental data presented below.

3.1 S-unimodal Maps

The two key results we require were established, respectively, by Singer [28] and Guckenheimer [12] only a little more than a decade ago. Here, we sketch the logical relation of the original authors' general theorems to the particular consequences that we find so important for our applications. It should be hastily added that many other engineers, physicists, and applied mathematicians have already recognized the importance of these and allied results. Thus a variety of deeper and more comprehensive tutorial re-workings are now available in text books, that of Collet and Eckmann [9] being particularly relevant to this discussion.

Singer and Guckenheimer stated their results for a very particular class of functions that "preserve" the unit interval \( I \triangleq [0, 1] \). We will first present the results for this "normalized" class of functions and then discuss more practically useful extensions via change of coordinates.

3.1.1 Stability

Before proceeding, it is helpful to fix terminology. In the engineering literature [13] one defines the stability of a set, \( \mathcal{P} \), to mean that for every open set, \( \mathcal{O} \), such that \( \mathcal{O} \supset \mathcal{P} \), there exists an open subset, \( \mathcal{S} \), with \( \mathcal{O} \supset \mathcal{S} \supset \mathcal{P} \), such that all trajectories of initial conditions in \( \mathcal{S} \) remain forever in \( \mathcal{O} \). Asymptotic stability adds the property that \( \mathcal{P} \) is an attracting set — i.e., constitutes the forward limit set — for all initial conditions in some open cover. In contrast, Singer and Guckenheimer include both properties within the definition of stability. Moreover, Guckenheimer includes the "one-sided" case — i.e. where the set \( \mathcal{P} \) is allowed to lie on the boundary of \( \mathcal{O} \) — as well. Similarly, in the engineering literature one usually refers to the set of points that converge to an asymptotically stable fixed point as its domain of attraction, while in contemporary mathematical treatments this set is more often known as the stable manifold.

For our audience, it seems preferable to retain the engineering usage. Thus, an asymptotically stable fixed point lies in the interior of an open interval, which interval is itself contained in the domain of attraction of that fixed point. Of course, the same terminology applies automatically to the stability properties of higher period orbits since these are fixed points of some iterate of a map.

What stability behavior do we desire? For practical reasons, engineers will not be interested in so a weak notion of convergence as specified by the "one-sided" case. In practice, the local notions of stability and asymptotic stability that guarantee qualitatively indistinguishable behavior for all initial conditions in a sufficiently small neighborhood of a fixed point have often
proven useful. Indeed, they have the great virtue of admitting a simple numerical test. However, we have found in our juggling work [4, 5] that the domain of attraction of asymptotically stable fixed points is often too small to be physically observed. Thus we have found the need for some means of assuring a sufficiently large domain of attraction. Short of global asymptotic stability which may be too strong to hope for or prove, it should suffice that the complement of the domain of attraction is a "small set" in the state space.

**Definition 1** A periodic orbit is essentially globally asymptotically stable if it is asymptotically stable and has a domain of attraction whose complement in the state space has zero measure.

### 3.1.2 Normal S-Unimodal Maps

A key sufficient condition for the results that follow is a sign condition on the Schwarzian derivative [15] of a real function \( f \in C^3[\mathbb{R}] \), defined as

\[
Sf(x) \triangleq \frac{f''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2.
\]

Singer, who first noted the utility of this condition, considered a slightly larger class of functions than those specified by the following definition.

**Definition 2** (Guckenheimer [12]) A function, \( f \in C^3[I] \), is said to be normal S-unimodal if it satisfies the following conditions:

1. \( f(0) = f(1) = 0 \)

2. \( f \) has a single local maximum, \( c \in I \). The function is strictly increasing on \([0, c]\) and strictly decreasing on \([c, 1]\). \( f''(c) < 0 \).

3. The Schwarzian Derivative of \( f \) is negative: for all \( x \in I - c \), \( Sf(x) < 0 \).

Singer’s result established the simplicity of converging orbits for this class of functions.

**Theorem 1** (Singer [28], [12]) If \( f \) is a normal S-unimodal function then it has at most one attracting periodic orbit. Moreover, the critical point, \( c \), lies in the interior of the connected component of the domain of attraction including some member of that periodic orbit.

**Corollary 3.1** Let \( f \) be a normal S-unimodal function with an asymptotically stable periodic orbit. Then there is an open interval, \( U \), containing both \( c \) and a member of that periodic orbit with the property that every orbit in the domain of attraction eventually enters \( U \).
Proof: Since our definition of asymptotic stability excludes the one-sided situation, we know according to Theorem 1 that the domain of attraction includes an open interval, \( \mathcal{U} \), containing both \( c \) as well as some point, \( p \), in the asymptotically stable periodic orbit. Since sufficiently small neighborhoods of \( p \) are contained in \( \mathcal{U} \) too, any \( x \in \mathcal{I} \) in the domain of attraction has a trajectory which eventually enters \( \mathcal{U} \).

\( \Box \)

Guckenheimer achieved a topological classification of the dynamical systems defined by iterated normal S-unimodal maps that he used to characterize their sensitivity to initial conditions in the framework of measure theory. In so doing he was led to characterize the topology of initial conditions whose orbits "stay away" from the critical point, \( c \).

**Proposition 3.2 (Guckenheimer [12])** Let \( f \) be a normal S-unimodal map and let \( \mathcal{U} \) be an open interval containing \( c \). If \( f | \mathcal{I} \setminus \mathcal{U} \) has no attracting periodic orbits then the set

\[
E_\mathcal{U} \triangleq \{ x | f^n(x) \in \mathcal{I} \setminus \mathcal{U} \text{ for all } n \geq 0 \}
\]

is totally disconnected.

He was led, as well, to determine the measure of such sets that "stay away" from \( c \).

**Theorem 2 (Guckenheimer [12])** Let \( f \) be a normal S-unimodal map and let \( \mathcal{U} \) be a neighborhood of \( c \). If \( E_\mathcal{U} \) is totally disconnected then it has Lebesgue measure zero.

We obtain the key result by combining Corollary 3.1 with Guckenheimer's Theorem and Proposition to get

**Corollary 3.3** Let \( f \) be a normal S-unimodal function. Then any asymptotically stable periodic orbit is essentially globally asymptotically stable as well.

### 3.1.3 Smooth S-Unimodal Maps

To what extent are the restrictive normal S-unimodal conditions necessary to the strong conclusions we have reported above? It is clear (and will be apparent at the end of the discussion in this section) that the Schwarzian condition is not necessary. Unfortunately, as it turns out, there do not yet appear to be any better answers to the question in the present day mathematical literature. Here, we will offer certain weak (and unsurprising) extensions that nevertheless bring the present application into compliance with the desired properties.

Control engineers are intimately familiar with the fact that stable linear systems maintain their stability properties under linear changes of coordinates. We now review the extension of "coordinate change" to a form suitable for nonlinear systems. A good reference for these ideas is provided in [10].
Definition 3 Given two (possibly unbounded) real intervals, \( A, B \), a continuous bijective function, \( h : A \rightarrow B \), is said to be a homeomorphism if \( h^{-1} \) is also continuous. If \( h \) and \( h^{-1} \) are smooth (continuously differentiable up to degree \( M \) ) then \( h \) is said to be a \( (C^M) \) diffeomorphism.

Let \( f \) map \( A \) into itself, and \( g \) map \( B \) into itself. The two maps are topologically conjugate if there exists a homeomorphism, \( h \), such that

\[
g \circ h = h \circ f.
\]

If \( h \) is also a diffeomorphism, then the two maps are differentiably conjugate. The map \( h \) is called a conjugacy or change of coordinates.

The reader may check that if \( h \) is a conjugacy between \( f \) and \( g \) then \( p \) is a fixed point of \( f \) (or its iterates) if and only if \( h(p) \) is a fixed point of \( g \) (or its iterates). Moreover, the iterates of \( f \) through some initial condition converge to a periodic orbit if and only if the iterates of \( g \) through the image of that initial condition converge to the image of the periodic orbit under \( h \). Thus, the stability properties of a map are preserved under change of coordinates. In particular, the conclusion of Singer's Theorem and Proposition 3.2 hold for any conjugate, \( g \), of a normal \( S \)-unimodal map. That is to say, there can be at most one stable periodic orbit, the complement of whose domain of attraction is a totally disconnected set.

Unfortunately, not all coordinate changes preserve zero measure. For example by adding the Cantor function [10, 14] to the identity map and then scaling, one obtains a homeomorphism of the unit interval to itself. While the Cantor set has measure zero in \( I \) [14, §16.5(b)], its image under this continuous map has positive measure [14, §19.3(d)]. Thus, the complement of the domain of attraction of a topological conjugate to a normal \( S \)-unimodal map, while totally disconnected, might still have measure approaching unity.

We have unhappily stumbled upon the mathematically unparadoxical but practically ambiguous discrepancy between the topological and measure theoretic versions of a "insignificantly small" set. For example, while the commonly accepted definition of a topologically "generic" property is that it hold on some open dense set, such sets can have arbitrarily small measure [10, §1.2]. Conversely, the previous paragraph offers an instance of a complement of an open dense set — the Cantor Set is closed and nowhere dense [14, §16.5(c)] — having measure arbitrarily close to that of its open cover. Of course, we might simply call this example "pathological" and agree to restrict coordinate changes to those which are finitely piecewise differentiable thus, dispensing of the Cantor Function. It seems simpler, safer, and sufficient for our present purposes to restrict attention to smooth coordinate changes.

Definition 4 A map of some (possibly unbounded) real interval to itself is said to be smooth \( S \)-unimodal if it is \( C^1 \)-conjugate to a normal \( S \)-unimodal map.

Lemma 3.4 ( [16, Lemma 3.1.1]) The image of a zero measure set under a \( C^1 \) map has zero measure.

---

\(^1\)We are indebted to R. Beals for calling this fact to our attention.
Corollary 3.5 Let $f$ be a smooth $S$-unimodal map. Then any asymptotically stable periodic orbit is essentially globally asymptotically stable as well.

Notice that by restricting consideration to the smooth conjugates we enjoy a situation wherein the “bad” set of points that stay away from the attracting periodic orbit is “small” both in the measure theoretic as well as the topological sense. Notice, as well, that among its other consequences, this result demonstrates that the Schwarzian derivative sign condition (since it is not invariant under coordinate changes) cannot be necessary.

3.1.4 Bifurcations of Unimodal Maps

Say that $f$ is a normal unimodal function if it satisfies the first two conditions of Definition 2 but not necessarily the third. Say that $g$ is a unimodal function if it is topologically conjugate to some normal unimodal function. Singer [28] provides an example of a normal unimodal function that is not normal $S$-unimodal for which the conclusions of Theorem 1 are invalid. Thus, while $S$-unimodality is not necessary, neither is mere unimodality sufficient for concluding that a local stability property is essentially global.

On the other hand, when we consider one-parameter families of unimodal maps, then certain strong and essentially universal (i.e. independent of the particular parametrized family) properties hold true. Namely we can expect predictable structural changes in the qualitative dynamics pertaining to almost identically related values of the parameter, entirely independent of the details of the particular family. Let $g_\mu$ be a unimodal map for each $\mu$ in some real interval, $\mathcal{M} \subset \mathbb{R}$. A particular value, $\mu_0$, is said to be a bifurcation point if there is no neighborhood of $\mu_0$ in $\mathcal{M}$ such that $g_\mu$ is conjugate to $g_{\mu_0}$ when $\mu$ is in that neighborhood. Intuitively, the qualitative behavior of the dynamics changes around a bifurcation point.

Generically, if $g_{\mu_0}$ has a fixed point, $p$, and $g'_{\mu_0}(p) = -1$ then $p$ is asymptotically stable for nearby $\mu$ on one side and it is unstable for nearby $\mu$ on the other side of $\mu_0$ in $\mathcal{M}$. In such a situation, $g_\mu$ has a new asymptotically stable period two orbit for nearby $\mu$ on the latter side of $\mu_0$. This is called a period doubling bifurcation [10, Thm. 1.12.7] and is a well known local result.

Now suppose that $\{g_\mu\}$ is unimodal family. If there is an interval of values $\mathcal{M} \supseteq (\mu_0, \mu_\infty)$ of $\mu$ for which $g_{\mu_0}(c) = c$ and $g_{\mu_\infty}(c) = 1$, then we shall say that $g_\mu$ is a full family [11]. A full family must exhibit an accumulating cascade of period doubling bifurcations: i.e., from an asymptotically stable period one orbit until $\mu_1$, to an asymptotically stable period two orbit until $\mu_2$, an asymptotically stable period four orbit until $\mu_3$, and so on. Thus, unimodal families give rise to theoretically determined global bifurcation diagrams. A typical such bifurcation diagram is displayed in Figure 4 as taken straight out of [9].

Unimodal period doublings have a universality as follows. Denote as $\mu_n$ those points in the bifurcation diagram where there is a bifurcation from length $2^{n-1}$ to $2^n$. Then the ratios $\frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}$ converge to some universal number $\delta = 4.66920...$ regardless of the family or the details of the parametrization. We will show later that our Gedankenrobot juggler satisfies these conditions and indeed, from the simulated bifurcations diagram we can verify this universal number.
3.1.5 Practical Summary

In the fortunate case of encountering a smooth $S$-unimodal return map, $f$, the job of determining the essential global limit behavior of its associated dynamics reduces to simple algebra and calculus. After finding the fixed points of the $N^{th}$ iterate of $f$ (algebra), we compute the magnitude of its first derivative (calculus) at a fixed point. If that magnitude is less than unity then we may expect all experiments performed upon the corresponding physical apparatus (with gains set to the appropriate values) to result in periodic steady state behavior exhibiting no more than $N$ distinct states.

If, moreover, we encounter a one-parameter family of smooth $S$-unimodal maps, and the family is full, then appropriate adjustments of the parameter will afford any conceivable stable periodic behavior. Eventually, when the period, $N$, gets to be a sufficiently large number, our ability to distinguish periodic from "chaotic" steady state behavior will be compromised by the imprecision of our measurements.

3.2 Applications to the Mirror Law Return Map

We now explore the implications of the preceding theory for our particular dynamics. In order to be directly applicable to our physical apparatus, we will use the models that include friction in the following sections. The case without friction can readily be recovered by setting $\zeta = 1$.

3.2.1 The Gedankenrobot

Recall the original map (10) for successive impact velocities

$$f(b) = \zeta b \left( 1 - \beta (b^2 - b^{*2}) \right),$$

defined on the interval

$$W = \left( -\sqrt{b^{*2} + \frac{1}{\beta}}, 0 \right) = (b_0, 0).$$

Recall from (11) that this function has a fixed point at $b_{fp} = -\sqrt{b^{*2} + \frac{1 - 1/\zeta}{\beta}}$ and a unique minimum at $b_c = \frac{1}{\sqrt{3}} b$ with $f(b_c) = \zeta \frac{2}{3} b_c^3$.

**Theorem 3** The mirror algorithm for the Gedankenrobot results in a successful vertical one-juggle which is essentially globally asymptotically stable as long as

$$0 < \beta < \frac{2/\zeta - 1}{b^{*2}}.$$

**Proof:** First note that $f$ is smoothly (in fact, linearly) conjugate to

$$g(x) = h \circ f \circ h^{-1} = \mu x (1 - x^2); \quad \mu = \zeta (1 + \beta b^{*2})$$

(16)
via the change of coordinates

\[ h(\hat{b}) = \hat{b} / \hat{b}. \]

We now show that \( g \) is a normal S-unimodal map. The conjugate, \( g(x) \), is defined on

\[ I = (0, 1) \]

with a unique maximum at \( c = \frac{1}{\sqrt{3}} \). Since \( g(c) = \frac{2}{3\sqrt{3}} \mu \), the positive invariance condition \( g : I \to I \) is satisfied for \( \mu \in \left( 0, \frac{3\sqrt{3}}{2} \right) \). Moreover, \( g(0) = g(1) = 0 \), and \( g \in C^3(I) \). Thus, condition 1 of Definition 2 is satisfied. Further, \( g \) has a single local maximum at \( c = \frac{1}{\sqrt{2}} \), is monotone up on \([0, c]\) and monotone down on \([c, 1]\). Note that \( g''(c) = -6\mu \zeta < 0 \). Thus, condition 2 of Definition 2 is satisfied. Finally, \( S g(x) = -\frac{6\mu}{\zeta^2} (1 + 6x^2) < 0 \) on \( I \).

This implies that \( f \) is a smooth S-unimodal map: according to Corollary 3.5, the fixed point \( \hat{b}_{fp} \) is essentially globally asymptotically stable as long as \( f' \) has magnitude less than unity. The result now follows since \( f'(\hat{b}_{fp}) = 3 - 2\zeta(1 + \beta b^2) \).

\[ \square \]

It is easy to verify that \( g_\mu \) is also a full family: For \( \mu \in (\mu_0, \mu_\infty) = (0, 0.3\sqrt{3}) \), we obtain \( g_{\mu_0}(c) = c \) and \( g_{\mu_\infty}(c) = 1 \). Now we know that after the fixed point \( \hat{b}_{fp} \) becomes unstable, the map \( f \) will exhibit period doublings that will eventually lead to chaotic behavior, as predicted before. This is confirmed in Figure 3 and Figure 6, which show the bifurcation diagram (obtained via simulation) for our specific Gedankenrobot juggler map without and with friction, respectively. The ratio \( \frac{\mu_{n+1} - \mu_n}{\mu_n - \mu_{n-1}} \) is evaluated for the first three bifurcations, \( n = 2 \), directly from the two figures, with an accuracy of about 0.3 due to the large \( \beta \) stepsize. For the juggler without friction one obtains \( \delta = 4.9 \) and with friction \( \delta = 4.6 \). Both values are close to the expected limiting value for \( n \to \infty \) of \( \delta = 4.66920... \).

Given the settings \( \mu_{fric} = 0.16, \alpha = 0.7, \zeta = 0.9115, \) and \( \hat{b}^* = -125 \), one can predict from \( f'(\hat{b}_{fp}) = -1 \) the first \( \beta \)-bifurcation values for the two cases as 6.4 and 7.6. Both values are confirmed accurately from the two figures, using again \( \beta = \frac{1+\alpha}{2} \kappa_{11} \).

### 3.2.2 The Two Degree of Freedom Case

We have now at hand very strong tools for analyzing many scalar discrete maps. At first sight, the fact that these results hold only for scalar dynamics seems to be very limiting, as most real world systems possess more than one degree of freedom. However, there are some ways in which the new insights can be applied to more complex systems, even though the mathematical results corresponding to Theorems 1 and 2 are not (yet) available for higher dimensional maps.

For a variety of systems one can speculate about ways to apply the scalar results. Notice that to make these ideas rigorous, all of them require additional analytical work. The first, and simplest class of higher order systems can be completely decoupled into independent scalar systems via a suitable nonlinear coordinate transformation. After decoupling, the scalar results
can be applied to the independent subsystems. The second class of systems can be shown to be decoupled at steady state. Here we have to identify and isolate again via a change in coordinates, a stable one dimensional submanifold. A third class of systems might be coordinate transformed into a system that has a dominant one degree of freedom subsystem with weak coupling to the rest of the degrees of freedom. This is analogous to approximating a high order or infinite dimensional linear system with a lower dimensional subsystem which captures the dominant dynamics.

Now we will show that our two degree of freedom juggler belongs to the second class of systems. This case is very simple in that not even a coordinate transformation is necessary to exhibit the scalar system at steady state.

Even though we can’t even write down the discrete map corresponding to the closed loop system in two dimensions, it is easy to show, that along the desired horizontal position, \( b_1 = b_1^* \), the vertical dynamics are identical to (10). First, notice that the puck dynamics in isolation along the vertical line must be identical to (8). Therefore, we only have to show that the mirror law (15) along \( b = b_1^* \) and \( \dot{b}_1 = 0 \) is identical to the mirror law of the Gedankenrobot (6).

For \( (b_1 - b_1^*) = (\dot{b}_1 - \dot{b}_1^*) = 0 \), the mirror law (15) becomes

\[ \theta_r(t) = -\kappa_1(w)\theta_p. \]

Similar to the derivation in Section 2.1 impacts can only occur when \( \theta_r = \theta_p = 0 \) and thus \( b_2 = 0 \) as long as \( \dot{b}_2 \in W \). Furthermore, at the impact position, the linear robot impact velocity is identical to Gedankenrobot case with

\[ u_2 = b_1^* \dot{\theta}_r = -b_1^* \kappa_1(w) \dot{\theta}_p = -b_1^* \kappa_1(w) \frac{\dot{b}_2}{b_1^*} = -\kappa_1(w)\dot{b}_2, \]

where again, \( \theta_p = \tan \frac{b_2}{b_1^*} \).

Since all impacts occur at \( b_2 = 0 \), the time of flight is

\[ t = \frac{b_2'}{\dot{b}_2'} \left( \frac{1}{\gamma_{wp}} + \frac{1}{\sqrt{\gamma_{wp} \gamma_{dn}}} \right) \]

where \( \dot{b}_2' \) is the puck velocity just after impact. In the case of no friction, \( \zeta = 1 \), this simplifies to \( u_1 = \frac{2}{\gamma_{wp}} \).

The derivation of the discrete impact map is from now identical to the one in Section 2.1, leading to a map identical to (10) where \( \dot{b}_2 \) replaces \( \dot{b}_1 \).

Now we can verify per computer simulation, that as predicted, the two degree of freedom juggler along \( b_1^* \) behaves just like the Gedankenrobot. The bifurcation diagrams of both systems are identical and provide excellent means for that purpose. Compare Figure 5 with Figure 7 for the systems without friction to verify our claims. The same result is obtained when comparing the respective systems with friction, which is omitted here for brevity.
Figure 4: Bifurcation Diagram for $f_{\mu} = 1 - \mu x^2$ as Shown in Collet and Eckmann
Figure 5: Gedankenrobot without Friction: Simulated Bifurcation Diagram
Figure 7: The Two Degree of Freedom Juggler without Friction: Simulated Bifurcation Diagram
Figure 6: Gedankenrobot with Friction: Simulated Bifurcation Diagram
4 Experimental Results

In this section we present experimental data to validate the models developed in Section 2 and the theoretical predictions by comparisons with simulated data from Section 3.2. In Section 4.1 devoted to the Gedankenrobot, we will use the theoretical insights presented in Section 3.1 for scalar return maps to predict the dynamical behavior of the physical apparatus. The correspondence between simulated and experimental data is gratifying. We are able to predict and verify experimentally the transients of the stable fixed points as well as higher period stability properties, specifically bifurcations to stable period two and period four orbits. In Section 4.2 we back with experiments our speculations from Section 3.2.2 about the applicability of the theory to the two degree of freedom juggler.

4.1 Gedankenrobot Experiments

![Diagram of Gedankenrobot Implementation]

Figure 8: The Gedankenrobot Implementation

How can we implement the one degree of freedom Gedankenrobot on our planar juggler? We have seen in Section 3.2.2 that the closed loop dynamics of the planar juggler along the desired horizontal impact position, δ², are captured exactly by the Gedankenrobot impact map (10). If our system were completely noise and error free, and the horizontal puck position were controlled perfectly, the one dimensional Gedankenrobot dynamics could be observed directly along a vertical line on the planar juggler. As both assumptions are unrealistic, we exclude the possibility of any horizontal errors and thus preclude cross-coupling from the horizontal to the vertical
dynamics by physically restricting the puck motion to the vertical line. This is accomplished by
spanning two wires across the juggling plane as shown in Figure 8, forming a guiding rail for
the puck along the desired horizontal position \( b_1 = b_1^* \).

We have shown in the previous section that the local dynamical behavior is essentially
global. The data in Figure 9 confirm that the transients can be predicted by recourse to local
linear analysis of the scalar impact map. Evaluating the derivative of (10) at the fixed point
(12) for the four gain settings \( \kappa_{11} = 3/5/7/9 \cdot 10^{-5} \) shown in the figure, we predict locally
an overdamped, critically damped, underdamped and an unstable response, respectively. This
behavior is confirmed even from large initial conditions ("globally") on the juggling apparatus.
When inspecting the transient for the last gain setting \( \kappa_{11} = 9 \cdot 10^{-5} \) closely we see that it
maintains a small but steady oscillation between two very close impact velocities. By examining
the local derivative of the impact map (12) we predict instability starting at \( \kappa_{11} = 8.99 \cdot 10^{-5} \).
Moreover, according to theory, this value is a bifurcation point to a stable period two orbit. We
will see shortly that this is indeed the case.

The fixed point in the presence of friction (11) depends on the gain setting \( \kappa_{11} \). In Figure 9
we confirm that the predicted steady state values for the vertical impact velocities are reached
within less than 3% error. The following table summarizes the results concerning the transient
and steady state responses up to this point in detail.

<table>
<thead>
<tr>
<th>Gain ( \kappa_{11} )</th>
<th>Fixed Point (predicted)</th>
<th>Fixed Point (measured)</th>
<th>Local Derivative</th>
<th>Global Transient</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 ( \cdot 10^{-5} )</td>
<td>108</td>
<td>105</td>
<td>+0.45</td>
<td>overdamped</td>
</tr>
<tr>
<td>5 ( \cdot 10^{-5} )</td>
<td>115</td>
<td>113</td>
<td>-0.03</td>
<td>crit. damped</td>
</tr>
<tr>
<td>7 ( \cdot 10^{-5} )</td>
<td>118</td>
<td>117</td>
<td>-0.50</td>
<td>underdamped</td>
</tr>
<tr>
<td>9 ( \cdot 10^{-5} )</td>
<td>120</td>
<td>119</td>
<td>-1.00</td>
<td>unstable</td>
</tr>
</tbody>
</table>

When the feedback gain \( \kappa_{11} \) is increased further, we observe at a gain setting of about
\( \kappa_{11} = 9.5 \cdot 10^{-5} \) the period doubling that is predicted by theory and the simulations in Section 3.2
at a value of \( \kappa_{11} = 8.99 \cdot 10^{-5} \). The experimentally measured bifurcation diagram shown in
Figure 10 coincides with great accuracy with the model prediction in Figure 6 up to the second
bifurcation which leads to period four orbits. These data were acquired in an analogous fashion
to the simulated plot. The puck was dropped at a height corresponding to \( \dot{b} = -100 \) in/sec.
We discarded the first 20 impacts to assure steady state juggling and logged the following 50
impacts. For the last four gain settings we logged the following 100 impacts since the spread of
impact velocities increased. This was repeated for the \( \kappa_{11} \) gain range in \( 0.25 \cdot 10^{-5} \) increments.

The maximum achievable gain \( \kappa_{11} \) was limited to \( 12.25 \cdot 10^{-5} \) for the following reasons.
First, with increasing gain, one impact velocity of the periodic orbit gets closer and closer to
zero. Note that this is equivalent to the previous impact getting closer and closer to the minimum
allowable value of \( \dot{b} = -159.5 \) in/sec. This can be verified from the discrete impact map (10).
Second, as the feedback gain increases, the magnitude of the perturbations introduced by the
juggler increase as well. In addition, for very low impact velocities we experience relatively large
puck position sensing inaccuracies which can result in unacceptable errors in impact position.
Together, these effects invariably cause some of the 100 impacts to make improper contact, like
multiple bounces, or even fall off the plane, which invalidates the runs.
For illustration of a stable period two orbit, that is, an unstable fixed point of \( f \), but a stable fixed point of \( f \circ f \), we show a transient juggle in Figure 11 for the gain setting \( \kappa_{11} = 11 \cdot 10^{-8} \). We start at the (now unstable) fixed point 120.8 in/sec computed from (11) and observe convergence to the stable period two orbit.

Unfortunately the juggler performance degrades for higher gain settings before we can actually see chaotic behavior. However, there is some indication of the existence of a period four orbit in Figure 10 for the last gain setting \( \kappa_{11} = 12.25 \cdot 10^{-6} \). Indeed if we display that trajectory over time, or here more correctly, over impacts, we find emerging — and, due to perturbations again disappearing — discernible period four orbits. An example of such a period four orbit is shown in Figure 12.

### 4.2 Planar Juggler Experiments

We now remove the physical constraints on the juggling plane and perform the same experiments. We argued in Section 3.2.2 that we can still apply the analytical insights and predictions from the Gedankenrobot: if the horizontal position errors are kept small by the control law. This can be verified in Figure 13 which displays an experimentally measured bifurcation diagram for the planar juggler. Again, the puck is dropped at a drop off position \( b_1^* \) and with an initial height which results in an initial vertical impact velocity of about \(-100\) in/sec. After the first 20 impacts have passed, we log the next 100 impacts shown in the plot. We start with an initial value of \( 6 \cdot 10^{-6} \) and record a run until \( \kappa_{11} = 12.5 \cdot 10^{-5} \).

This bifurcation plot is almost identical to that of the Gedankenrobot juggler in Figure 10. Due to the larger perturbations present, the spread of the data is considerable larger. However, we can still identify a bifurcation to a stable period two orbit after the value of \( \kappa = 10 \cdot 10^{-5} \), which is close to the value \( \kappa_{11} = 8.99 \cdot 10^{-5} \) predicted from the theory of the simpler system.

The main reason for the degraded quality of this bifurcation plot at period two orbits versus the previous one are the added disturbances due to horizontal position errors. At the relatively high gain settings, even small errors in the vertical hitting position, and thus the hitting angle at one impact, translate into large horizontal position errors at the next impact. Even though the horizontal position regulation is still stable, convergence at high vertical feedback gain is underdamped due to crosscoupling and therefore slower. The resulting effect on the bifurcation plot will be even better understood if we take another look at Figure 11. Even under ideal experimental conditions, the period two transients are relatively slow and thus it takes a long time to recover from the aforementioned perturbations. However, it is encouraging that, in spite of these disturbances, we can still observe the predicted steady state values (in the presence of friction), and the first bifurcation to period two oscillations.
Figure 10: The Gedankenrobot Bifurcation Diagram: Experimental Data
Figure 11: A Transient Toward Period Two: Experimental Data
Figure 12: A Period Four Sequence: Experimental Data
Figure 13: The Bifurcation Diagram for the Unconstrained (2dof) Juggler: Experimental Data
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References


