Exact Robot Navigation
in Geometrically Complicated
but Topologically Simple Spaces

A paper submitted for presentation in the
1990 IEEE International Conference on Robotics and Automation

Elon Rimon and Daniel E. Koditschek

Center for Systems Science
Yale University, Department of Electrical Engineering

October 15, 1989

1This work was supported in part by the National Science Foundation under grants DMC-8505160,
and a Presidential Young Investigator Award held by the second author.
CONTENTS

Contents

1 Introduction ................................................. 3
  1.1 Recasting the Navigation Problem: The Longer Term Program of Research ........... 4
    1.1.1 Other Versions of the Navigation Problem ........................................ 4
    1.1.2 Our Results .................................................. 5
  1.2 Problem Statement .......................................... 6
    1.2.1 Workspace and Obstacles ......................................... 6
    1.2.2 Navigation Functions and Their Invariants ................................. 8
  1.3 The Available Information ...................................... 10

2 Construction of the Transformation ................................ 12
  2.1 Forests of Stars ............................................ 12
    2.1.1 Star-Shaped Sets and Obstacles ........................................... 12
    2.1.2 Simple Quadratic Trees of Stars .......................................... 13
  2.2 The Transformation ........................................... 13
    2.2.1 Some Preliminary Definitions ........................................... 15
    2.2.2 The Definition of the Transformation ..................................... 15
    2.2.3 Discussion of the Transformation ....................................... 18
    2.2.4 The Geometrical Information Required ..................................... 19

3 Correctness of the Construction ................................... 22
  3.1 $f_\Lambda$ is an Analytic Diffeomorphism if Its Jacobian is Non-Singular .......... 22
  3.2 The Jacobian of $f_\Lambda$ is Non-Singular Away from the “Sharp Corners” ....... 23

4 Counting the Floating Point Operations ................................ 26
  4.1 The Computation of the Parameters in $\varphi$ .................................... 26
  4.2 The Computation of $\nabla \varphi$ ........................................... 27

5 Simulation Studies .............................................. 29
A  A "Calculus" for Implicit Representations  

B  Some Details

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1 Details of the Construction</td>
<td>39</td>
</tr>
<tr>
<td>B.2 A Proof of the Theorem</td>
<td>42</td>
</tr>
<tr>
<td>B.3 The Jacobian of $f_\lambda$ is Non-Singular</td>
<td>45</td>
</tr>
</tbody>
</table>
Abstract

A navigation function is an artificial potential energy function on a robot configuration space which encodes the task of moving to an arbitrary desired destination without hitting any obstacle. In particular, such a function possesses no spurious local minima. In this paper we construct navigation functions on forests of stars: geometrically complicated configuration spaces that are topologically indistinguishable from simple discs punctured by disjoint smaller discs, representing "idealized" obstacles. For reasons of mathematical tractability we approximate each "obstacle" by a Boolean combination of linear and quadratic polynomial inequalities (with "sharp corners" allowed), and use a "calculus" of implicit representations to effectively represent such obstacles. We provide evidence of the effectiveness of this "technology" of implicit representations in the form of several simulation studies illustrated at the end of the paper.

1 Introduction

This paper presents the last in a series concerning exact (and provably correct) navigation in perfectly known environments possessed of a particularly simple topology — the "sphere worlds". In previous work, we have shown that only the topology matters [7, 10]. Moreover, we have constructed an explicit "navigation-function" algorithm for the simplest geometric instance of the sphere world topology [17, 10]. Subsequently [15, 16], we have provided an algorithm for a much richer variety of geometric detail in that topological equivalence class. This paper presents a methodical recipe for handling what we surmise (but have not formally proven) to be "almost every" possible geometric instance of these sphere worlds. Our present results thus conclude the first stage of a larger research program, attempting to recast the robot navigation problem as the search for a control law that uses an effective parametrization of the robot's à priori knowledge concerning its environment to move in cluttered spaces.

The results of this paper offer persuasive evidence that parametrizations which respect the configuration-space topology are effective with respect to both controller design and geometric complexity. Our controller — a refinement of the well known artificial potential field design [5] — guarantees convergence of the closed-loop robotic system to the desired destination from "almost all" initial configurations (no smooth time invariant controller can do better than this [10]), and guarantees that no collisions with any obstacle will occur along the way. Our recipe for handling geometric details relies upon a "calculus" for implicit representations of semi-analytic sets worked out in the Russian literature [20]. ¹ The assumption of perfect à priori information is obviously unrealistic and its relaxation comprises the second stage of our program. In this context, there arise problems of classification and adaptation whose proper formulation we are only beginning to understand.

The paper is organized as follows. This section continues with a brief literature review including an assessment of our results to date. It concludes with a formal statement of the problem at hand, and a specification of the assumptions concerning the available information. In the next section we present the machinery of our algorithms: we define the class of forests of

¹We would like to thank Vadim Shapiro and Herb Voelcker for alerting us to the existence of this literature.
stars, and present an explicit one-parameter family of "purging" transformations, each mapping a given forest onto the same forest denuded of its leaves. In Section 3 we outline the proof correctness, that constitutes the chief formal contribution of the paper (Theorem 1). For reasons to be made clear below, the availability of such a transformation solves the navigation problem as posed here. In Section 4 we discuss the computational complexity of this procedure, which turns out to be \textit{polynomial} in the "geometrical complexity" and, in the worst case, \textit{exponential} in the degrees of freedom. Finally, in Section 5, some simulation studies suggest the practicability of this methodology. We relegate to the appendices a discussion of the Russian "calculus" of implicit representations mentioned above, along with some computational details.

1.1 Recasting the Navigation Problem: The Longer Term Program of Research

A kinematic chain, actuated by idealized bounded torque motors, is allowed to move in a cluttered and imperfectly known workplace. Contained within the robot configuration space is the \textit{free space}, $F$ - the set of all robot placements which do not involve intersection with any of the "obstacles" cluttering the workplace. We are interested in the following loosely defined problem. Find a parameterized representation of the free space and desired destination. Find a map from the parameter space into a set of bounded-torque feedback controllers such that for each parameter value the closed loop robotic system (resulting from application of the controller) -- a vector field on the robot phase space -- moves the robot phase space -- moves the robot to the destination without hitting the parameter-determined configuration space obstacles. Find a procedure for adjusting the parameter values in conjunction with new sensory data such that the controller brings the robot to the desired destination, or parametric adjustment ceases with the decision that the desired state is not reachable.

We have rigorously formulated and solved only a particular piece of the problem. Namely, we are presently able to map a parameterized class of configuration spaces to bounded-torque controllers. In order to suggest how this piece of work points toward the larger research program it seems worth pausing first to review the general robot motion-planning literature.

1.1.1 Other Versions of the Navigation Problem

The purely geometric problem of constructing a path between two points in a space obstructed by sets with arbitrary polynomially described boundary given perfect information has been completely solved by Schwartz and Sharir[18]. Moreover, a near optimally efficient solution has been offered by Canny[3] for this class of problems as well. Unfortunately, it is not presently possible to use these methods "incrementally". Given new information about the world, the computation of a new solution must, in principle, start from the beginning. Moreover, the generation of paths does not address the need for a controller.

The idea of using "potential functions" for the specification of robot tasks with a view of the control problems in mind was pioneered by Khatib[5] in the context of obstacle avoidance. Fundamental work of Hogan[4] in the context of force control further advanced the interest in
1.1 Recasting the Navigation Problem: The Longer Term Program of Research

this approach. A similar methodology has been developed independently by Arimoto in Japan [1], and by Soviet investigators as well [12]. This version of the navigation problem has the virtue of admitting parametric information about the underlying configuration space directly into the controller structure with the result that small changes in present information might result in small changes in the controller. Unfortunately, the use of potential functions has been (until now) a poor heuristic with respect to the global path-planning problem. There are well known and frequent failures due to the appearance of spurious local minima that “trap” the robot away from its nominal destination. In addition, the heuristic approach has paid the price of unbounded controller inputs for the guarantee of obstacle avoidance.

Finally, Lunetsky [11] has pioneered the investigation of robot navigation using no a priori information. Although achieving wide success in two-dimensional configuration spaces, his results seem to suggest that only very special higher dimensional problems will yield practicable algorithms [19]. Moreover, his techniques provide no means of accumulating better and better information about the configuration space as navigation proceeds.

1.1.2 Our Results

In a recent paper [10], we propose a specification for a class of real-valued functions on the robot freespace, $\mathcal{F}$ — the class of navigation functions — which solves both the control problem and the path generation problem (up to the limits that the topology of $\mathcal{F}$ allows). We constructed a navigation function on any Euclidean sphere world (an $n$-dimensional disc in $\mathbb{E}^n$ punctured by any finite number of smaller disjoint $n$-dimensional discs) as a “model” of its topologically deformed class [10]. In a second paper [16], we construct navigation functions on any star world: a compact star-shaped set in $\mathbb{E}^n$ punctured by a finite number of smaller disjoint star-shaped sets. To achieve this construction on such a “complicated” space, $\mathcal{F}$, we “pull back” a navigation function on the simpler sphere world “model”, via a “change of coordinates”. We have supposed that perfect information about these spaces has been furnished in the form of the implicit representation of the star obstacles’ boundaries.

Unfortunately, our star world construction requires the obstacles to be disjoint analytic manifolds with boundary. In particular, obstacles with “sharp corners” are excluded. Moreover, the restriction to star-shaped sets (while including, for instance, all convex sets) does not seem sufficiently expressive of the geometric complexity arising in realistic robotic obstacle course. Finally, the assumption that a priori information will take the form of a global implicit representation of each obstacle seems dubious.

We address all these problems here. First we allow obstacles with sharp corners, and devise a means of making the gradient of the ultimate cost function smooth and globally bounded away from the sharp corners. Second, we dramatically increase the “geometric expressiveness” of the method by resort to trees of stars. These are obstacles built from intersecting star shapes, such that if one puts an edge between the centers of each pair of intersecting stars, the resulting graph is a set of disjoint trees. In the robotics context, spaces whose “obstacles” are comprised of intersecting stars arise naturally as the free configuration space for some (simple) one-link robotic situations (for example, a disc robot moving in a room cluttered with star-
shaped objects). Finally, we propose a methodology for describing a very rich variety of shapes, generated by a catalog of "building blocks" — planar and quadratic polynomial inequalities in $E^n$ — via a "calculus" of implicit representations. A representation of the shape at hand can be constructed with arbitrary precision in terms of the catalog's items, then, using the "calculus", the implicit representation of the resulting shape is automatically found. For example, in the star worlds, it might be preferable to approximate a star obstacle as a polyhedron in $E^n$ rather than as a polynomial of some high degree.

We suspect that obstacles in $E^n$ comprised of intersecting star-shaped sets can approximate the entire class of topologically deformed sphere worlds. Moreover, it is plausible that a tree-like decomposition of any topologically deformed disc into intersecting star-shaped sets could be found. However, we currently do not know how "far away" the sphere-world equivalence class lies from the most general realistic problem — the class of free configuration spaces mentioned above which arise when a general kinematic chain operates in a cluttered environment. Thus, we have solved a greatly restricted but rigorously posed version of the loosely defined problem above. By adopting the potential field framework we explicitly address the generation of feedback controllers. Yet our construction solves the geometric path planning problem exactly: convergence to the desired destination is guaranteed up to the constraints imposed by the topology of the free space. As an automatic consequence, the solutions are explicitly parameterized by the available geometric information. Since new data can be immediately assimilated into the existing constructions with very little effort, as long as the topology of the currently known environment doesn't change, this approach connects naturally to a limited (but precisely defined) portion of the adaptation problem for partially known environments [6].

1.2 Problem Statement

We will start by defining the workspace and the obstacles. Then we will define the class of navigation functions and show that they are invariant under analytic diffeomorphisms. Given a subset $S \subset E^n$, $\overset{\circ}{S}$ denotes its interior, $\overline{S}$ its closure, and $\partial S$ its boundary. A closed subset of a topological space is said to be "thin" (nowhere dense) when its interior is empty.

1.2.1 Workspace and Obstacles

The robot workspace, $W$, is a connected and compact $n$-dimensional subset of $E^n$, which is an analytic manifold with boundary. An obstacle, $O_j$, is the interior of a set in $E^n$ of the same type, whose closure is in the interior of the workspace, $\overline{O_j} \subset W$, and which is disjoint from every other obstacle.

$$\overline{O_i} \cap \overline{O_j} = \emptyset \quad 1 \leq i < j \leq M.$$  \hfill (1)

The free space,

$$F \triangleq W - \bigcup_{j=1}^{M} O_j.$$
remains after subtracting the obstacles from the workspace. Thus, \( \mathcal{F} \) is a compact subset of \( E^n \) punctured by \( M \) disjoint \( n \)-dimensional "holes" whose boundaries are analytic manifolds. It will prove convenient to refer to the complement of \( \mathcal{W} \) in \( E^n \) as the zeroth obstacle. In the special case in which the robot workspace and each obstacle removed from it is an \( n \)-dimensional disc in \( E^n \), the resulting free space,

\[
\mathcal{M} \triangleq \mathcal{W} - \bigcup_{j=1}^{M} \mathcal{O}_j,
\]

is an \( n \)-dimensional sphere world with \( M \) obstacles.

In the robotics context, even in the simplest (realistic) situations, the configuration-space obstacles have "sharp corners". The following definition extends the notion of an obstacle to include this possibility. A set \( \mathcal{O}_i \) in \( E^n \) is \emph{semianalytic} if it can be expressed as a finite Boolean combination (via the set operations \( \cup, \cap, -, \cdot \)) of sets of the form

\[
\left\{ q \in E^n : \beta_{ij}(q) \leq 0 \right\} \quad \text{for} \ j \in \{1, \ldots, l_i\},
\]

where \( \beta_{ij} : E^n \to \mathbb{R} \) are analytic \cite{13}. Roughly speaking, two surfaces meet transversally if they are not "tangent" to each other at any of their intersection points.

**Definition 1** An obstacle with corners, \( \mathcal{O}_i \), is a semi-analytic set in \( E^n \) comprised of a finite union and/or intersection of \( n \)-dimensional analytic manifolds with boundary, \( \mathcal{O}_{ij} \) for \( j \in \{1, \ldots, l_i\} \), such that

1. the resulting \( \mathcal{O}_i \) is a connected and compact \( n \)-dimensional topological manifold;
2. the boundaries of any two intersecting submanifolds, \( \partial \mathcal{O}_{ij} \) and \( \partial \mathcal{O}_{ik} \), meet transversally.

Using the Inverse Function Theorem, one can show that the transversality condition induces a cellular decomposition of \( \partial \mathcal{O}_i \) into \((n-1)\)-dimensional "patches" \( \mathcal{P}_{ij} \triangleq \partial \mathcal{O}_{ij} \cap \partial \mathcal{O}_i \) for \( j \in \{1, \ldots, l_i\} \), glued together along their respective boundaries. We denote by \( \mathcal{C}_i \) the boundary of these patches in \( \partial \mathcal{O}_i \),

\[
\mathcal{C}_i \triangleq \partial \mathcal{O}_i - \bigcup_{j=1}^{L_i} \mathcal{P}_{ij},
\]

the set of "sharp corners." Clearly, \( \mathcal{C}_i \) is closed and thin in \( \partial \mathcal{O}_i \). From now on, all our obstacles are assumed to have corners. For each obstacle, \( \mathcal{O}_i \), we require a knowledge of an "obstacle function": a continuous map \( \beta_i : E^n \to \mathbb{R} \), representing \( \mathcal{O}_i \) in the form

\[
\mathcal{O}_i = \{ q \in E^n : \beta_i(q) < 0 \} \quad \text{and} \quad \partial \mathcal{O}_i = \{ q \in E^n : \beta_i(q) = 0 \}. \tag{2}
\]

We require as well that each \( \beta_i \) be analytic on some open neighborhood about \( \overline{\mathcal{C}_i} - \mathcal{C}_i \) in \( E^n \). In general, \( \beta_i \) can be automatically constructed from the knowledge of an "obstacle function" for each of the constituent shapes \( \mathcal{O}_{ij} \) for \( j \in \{1, \ldots, l_i\} \), and the sequence of unions and intersections used to construct \( \mathcal{O}_i \), as we show in the Appendix.
Examples:

1. In the special case of the sphere world, these functions are given by

\[ \beta_0 \triangleq \rho^2 - \|q - q_0\|^2 \quad \text{and} \quad \beta_j \triangleq \|q - q_j\|^2 - \rho^2 \quad \text{for } j \in \{1, \ldots, M\}. \]

2. Any compact polyhedron is an obstacle with corners. Consider,

\[ \Omega_i \triangleq \left\{ q \in \mathbb{R}^n : \begin{array}{c}
(q - q_1) \cdot v_1 \leq 0, \\
\vdots \\
(q - q_m) \cdot v_m \leq 0
\end{array} \right\}. \]

As we discuss in Appendix A, an “obstacle function” for \( \Omega_i \) can be constructed as follows,

\[ \beta_i \triangleq \alpha_1 + \alpha_2 + \ldots + \alpha_m, \]

where

\[ \alpha_1 \triangleq \beta_{i1} + \beta_{i2} + \ldots + \beta_{im}, \]

\[ \alpha_2 \triangleq \sum_{j=2}^{m} \sqrt{\beta_{i1}^2 + \beta_{ij}^2} + \sum_{j=3}^{m} \sqrt{\beta_{i2}^2 + \beta_{ij}^2} + \ldots + \sqrt{\beta_{im}^2}, \]

\[ \alpha_k \triangleq (-1)^k \left( \sum_{j=k}^{m} \sqrt{\beta_{i1}^2 + \beta_{ij}^2} + \ldots + \sqrt{\beta_{i(m-k+1)}^2 + \beta_{im}^2} \right), \]

\[ \alpha_m \triangleq (-1)^m \sqrt{\beta_{i1}^2 + \beta_{i2}^2} + \ldots + \beta_{im}^2. \]

Indeed, according to Lemma A.1,

\[ \Omega_i = \beta^{-1}_i(-\infty, 0) \quad \text{and} \quad \partial \Omega_i = \beta^{-1}_i(0). \]

Moreover, \( \beta_i \) is continuous on \( \mathbb{R}^n \) and analytic on \( \mathbb{R}^n - \mathcal{C}_i \), where \( \mathcal{C}_i \) is the set of intersection points of two or more planes. Excluding degenerate cases -- for instance, two planes are identical -- \( \mathcal{C}_i \subset \partial \Omega_i \) is thin in \( \partial \Omega_i \).

1.2.2 Navigation Functions and Their Invariants

A function \( f \) is a homeomorphism if \( f \) is one-to-one and onto, and both \( f \) and its inverse are continuous. If both \( f \) and its inverse are analytic, then \( f \) is an analytic diffeomorphism.

Problem statement: Given a general \( n \)-dimensional free space, \( F \), each of whose obstacles is homeomorphic to the Euclidean \( n \)-disc, consider the pair \( (F, \mathcal{M}) \), where \( \mathcal{M} \) is an \( n \)-dimensional sphere world with the same (finite) number of obstacles. We seek a transformation \( f_\lambda \), defined on \( F \), satisfying

1. \( f_\lambda \) is a homeomorphism onto \( \mathcal{M} \);
2. away from a subset \( \mathcal{C} \subset \partial F \) which is thin in \( \partial F \), corresponding to “sharp corners”. 

\( f_\lambda \) is an analytic diffeomorphism.
1.2 Problem Statement

3. in each space there is a distinguished interior point — the destination point \( q_d \in \mathcal{F} \) and \( p_d \in \mathcal{M} \), such that \( f_\lambda(q_d) = p_d \).

We call this \( f_\lambda \) a modeling transformation.

The motivation for this problem is most simply provided by reference to the following definition of a navigation function and a method by which navigation functions can be induced on other spaces. A smooth real valued map \( \varphi : \mathcal{M} \to [0, 1] \) is a Morse function if at each of its critical points (a point \( q \) at which \( \nabla \varphi(q) = 0 \)) the Jacobian of the vector valued function \( \nabla \varphi(q) \) (the Hessian matrix of \( \varphi \)) is non-singular.

Definition 2 ([10], Def. 1 Revised) Let \( \mathcal{M} \) be an \( n \)-dimensional free space, which is connected and compact. A real valued map \( \varphi : \mathcal{M} \to [0, 1], \) is a navigation function if \( \varphi \)

1. is continuous on \( \mathcal{M} \) and analytic on \( \mathcal{M} - \mathcal{C} \);
2. has a unique minimum on \( \mathcal{M} \), at \( p_d \in \mathcal{M} \);
3. is Morse on \( \mathcal{M} - \mathcal{C} \);
4. attains its maximal value (uniformly) exactly on \( \partial \mathcal{M} \), the boundary of \( \mathcal{M} \);
5. has a bounded gradient on \( \mathcal{M} - \mathcal{C} \).

These conditions make \( \varphi \) a suitable artificial potential function. It is shown in [8, 9] that the bounded torque control law resulting from a navigation function defines a closed loop robotic system whose trajectories approach the destination point with zero velocity without intersecting obstacles, starting at almost any initial position in the robot configuration space. In general, this is the "strongest" convergence behavior that the topology of the underlying free space allows, as we have shown in [10]. Moreover, we have shown as well that smooth navigation functions exist on any smooth manifold with boundary — hence it makes sense to attempt explicit constructions in specific cases. In particular, we have shown how to do so on any sphere world.

The following Proposition, which obtains from application of the chain rule, for example, as discussed in [10], suggests that once a navigation function is constructed on a "model" space, \( \mathcal{M} \), it induces navigation functions on many other spaces via function composition.

Proposition 1.1 ([10], Proposition 2.6) Let \( \varphi : \mathcal{M} \to [0, 1] \) be a navigation function on \( \mathcal{M} \), and let \( h : \mathcal{F} \to \mathcal{M} \) be a modeling transformation. Then

\[
\varphi \triangleq \varphi \circ h,
\]

is a navigation function on \( \mathcal{F} \).
Thus, since we already know how to construct navigation functions on any sphere world, if a suitable sphere world model, $\mathcal{M}$, and an analytic diffeomorphism, $h$, can be found, we obtain a navigation function on $\mathcal{F}$ as well. We will use the Proposition to construct navigation functions on simple forests of stars (Def. 9 below). In particular, the new construction admits star worlds with "sharp corners", so that star obstacles can be described with arbitrary precision using simple shapes.

1.3 The Available Information

We are concerned with a free space $\mathcal{F}$ punctured by "obstacles" $\mathcal{O}_i$ for $i \in \{0, \ldots, M\}$. In a forest of stars (Def. 3 below), each obstacle in $\mathcal{F}$ is comprised of several intersecting stars. Denote the index set of all the stars in $\mathcal{F}$ by $\mathcal{I}$, and denote the various stars in $\mathcal{F}$ by $\mathcal{O}_i$, the same symbol denoting the tree-like obstacles, but with the index $i$ varying in $\mathcal{I}$. From hereafter we will regard each star in $\mathcal{F}$ as an obstacle by itself, with its own "obstacle function", $\beta_i$ for $i \in \mathcal{I}$, satisfying (eq. (2) above),

$$\mathcal{O}_i = \{ q \in \mathbb{R}^n : \beta_i(q) < 0 \} \quad \text{and} \quad \partial \mathcal{O}_i = \{ q \in \mathbb{R}^n : \beta_i(q) = 0 \}.$$ 

We assume project information, that is, for a given forest, $\mathcal{T}$, the intersection tree of the stars for each of the $M + 1$ tree-like obstacles, the obstacle function for each of the stars, as well as the star center points, $q_i$ for $i \in \mathcal{I}$, are known.

Moreover, for each star obstacle in $\mathcal{F}$, $\mathcal{O}_i$ for $i \in \mathcal{I}$, we assume the knowledge of constants, $\{E_{i1}, \ldots, E_{i4}\}$ and $F_{p(i)}$ (Sec. 2.2.4 below) describing bounds on geometrical features required in the construction of the transformation, and later in its proof of correctness. For example, $E_{i1}$ is an upper bound on the $i^{th}$ obstacle function, $\beta_i$, guaranteeing that

$$\beta_i^{-1}[0, 2E_{i1}] \cap \beta_j^{-1}[0, 2E_{j1}] = \emptyset,$$

for all $i, j \in \mathcal{I}$ such that $i \neq j$ and the stars $\mathcal{O}_i$ and $\mathcal{O}_j$ are not a "parent-son" pair. That is, the "$2E_{i1}$-thickened" boundaries of the stars still do not intersect. Since the closures of the various trees in $\mathcal{F}$, as well as the stars in the same tree which are not a "parent-son" pair, are disjoint, there is no problem with this data. Further, we will unhesitatingly make use of upper and lower bounds attained by various continuous functions on various compact sets without ever computing them explicitly. For example, for each star obstacle, $\mathcal{O}_i$, we require the knowledge of a lower bound on its minimal "radius" — the minimal distance from its center, $q_i$, to its boundary, $\partial \mathcal{O}_i$. In general, the extraction of these geometrical features from the knowledge of the obstacle functions may prove to be computationally intensive. However, in this report the obstacle functions are constructed from planar and quadratic polynomials, and explicit formulas giving bounds on the required features should be much easier to obtain.

Our scheme will be to transform the given forest, $\mathcal{F}$, onto a suitable star world, $\mathcal{F}_x$, which is derived from $\mathcal{F}$; and then to induce a navigation function on $\mathcal{F}$ using our previous construction of a navigation function, $\varphi_x$, on any star world [16]. In the construction of $\varphi_x$ we first derive a suitable model sphere world, $\mathcal{M}$, for the star world $\mathcal{F}_x$. Namely, we determine the center and radius of the $i^{th}$ sphere, according to the center and minimum "radius" of the $i^{th}$ star obstacle.
in $\mathcal{F}_s$. This in turn determines the model sphere world "obstacle functions", as well as the navigation function $\varphi_s$, in terms of the star world and the derived model sphere world.

In the robotics setting, the connectedness of the free space is not a realistic assumption. Certainly, the robot initial configuration in joint space determines a specific connected component of its free space, yet this might not include the destination point. At the present we rule out this possibility: our admissible spaces are homeomorphic to some sphere world, therefore connected.
2 Construction of the Transformation

In this section we define the class of spaces corresponding to forests of stars, and a one-parameter family of continuous functions, each of whose members is a candidate transformation of a forest onto its "purged" version. Composing several such transformations — the number of which is the depth of the deepest tree — yields a transformation of the original forest, $F$, onto a particular star world. Since we have already shown how to construct a transformation from any star world to a suitable model sphere world [16], and since we have shown how to construct a navigation function on any sphere world [10], the problem is solved according to Proposition 1.1.

We first define the class of forests of stars in Section 2.1; then, in Section 2.2, we define the purging transformation, the central contribution of this paper.

2.1 Forests of Stars

2.1.1 Star-Shaped Sets and Obstacles

A set $S \subset F^n$ with non-empty interior is star shaped (at $q_0$) if there exists a point $q_0 \in S$ such that for all $q \in S$, the line segment joining $q_0$ and $q$ is contained in $S$. Any star-shaped set is path-connected, and it can be shown that any open star-shaped set is homeomorphic to an open $n$-disc [2].

Consider an obstacle $O_i$ with a point $q_i \in O_i$. Given the implicit representation of its boundary in the form of the zero level set of some real-valued function, $\partial O_i = \beta_i^{-1}(0)$, there is a simple inner-product test on $\nabla \beta_i$ which checks that $\partial O_i$ is star-shaped with a center at $q_i$. Consider the ray starting at $q_i$ through a point $q$ in $\partial O_i$, $\alpha(s) = q_i + s(q - q_i)$ for $s \geq 0$. The instantaneous change in $\beta_i$ at this boundary point $q$ along the ray satisfies

$$\frac{d}{ds} |_{s=1} (\beta_i \circ \alpha)(s) = \nabla \beta_i(q) \cdot (q - q_i) > 0 \quad \text{for all } q \in \partial O_i.$$  \hspace{1cm} (3)

A more complete statement of this condition is given in Definition 9 in the Appendix. We call such obstacles strictly star shaped. The connection between the strictly star-shaped obstacles and star shaped sets is drawn in [14, Proposition 2.1]. Roughly speaking, a star-shaped set which is not strictly star shaped has a portion of some ray (from the center) lying in the star's boundary. Therefore, the collection of strictly star-shaped obstacles, each of whose members is closed and bounded, constitutes "almost all" bounded star-shaped sets. Our construction presumes that the various star-shaped obstacles satisfy condition (3), and from now on "star-shaped obstacle" will mean "strictly star-shaped".

Consider now an obstacle $O_i$ which is a union of several intersecting star obstacles. The arrangement of the stars in $O_i$ can be partially described by a graph, $(V, E)$, where $V$ is the set of vertices, and $E$ the set of edges (simple arcs that meet only at vertices). A tree is a connected graph with no (reduced) edge loops, and a forest is a set of disjoint trees.
2.2 The Transformation

Definition 3 Given a finite collection of star shaped obstacles, their connectivity graph is the graph \( (V, E) \) whose vertices, \( V \), are the star centers, and whose edges, \( E \), connect centers of stars having non-empty intersection.

A tree-of-stars obstacle is the finite union of intersecting star-shaped obstacles whose connectivity graph is a tree. A forest of stars is a free space, \( \mathcal{F} \), each of whose obstacles is a tree of stars.

In the special case in which each tree consists of one star, the resulting space is a star world [16]. Given a forest of stars, \( \mathcal{F} \), we label its stars by considering a tree at a time: for each tree, we designate one of its stars as the “root”, and then label the stars by proceeding along edge paths. The order in which we encounter the stars along a specific path determines a “parent–son” relation, and we denote the index of the parent of the \( i \)-th star by \( p(i) \). Each such path ends in a “leaf”, and we denote the index set of all the leaves in the forest which belong to trees consisting of more than one star by \( \mathcal{L} \). Finally, we denote the index set of all the stars in \( \mathcal{F} \) by \( \mathcal{I} \).

2.1.2 Simple Quadratic Trees of Stars

In this paper we construct navigation functions on a simplified subclass of forests of stars. First, we require that the constituent shapes of the various star obstacles (the shapes in the Boolean combination comprising \( O_i \) for \( i \in \mathcal{I} \)), be described by polynomial inequalities of degree at most 2. That is, the boundary of each star obstacle is partitioned into planar and quadratic patches, with “sharp corners” allowed. We call such obstacles \textit{quadratic}, and say that a free space \( \mathcal{F} \) is quadratic when all its obstacles are quadratic. Since the boundary of any star-shaped obstacle can be approximated with arbitrary precision using such patches, the quadratic forests of stars are “dense” in the class of all forests of stars. The second restriction requires that an admissible tree of stars have the following three properties: the center of each star, \( q_i \), is contained in its parent in the tree, each of the stars is connected to its parent via a unique patch, and this patch is “star shaped” with respect to \( q_i \). We call a tree-like obstacle having these three properties \textit{simple}, and say that a (quadratic) forest of stars is simple if all its tree-like obstacles are simple. A precise definition of these terms is given in Definition 10 in the Appendix. Figure 1 shows an example of a simple tree of stars, as well as unallowable situations. 2.1.2.

In this report we construct navigation functions on arbitrary simple quadratic forests, and from now on the term forest will mean simple and quadratic as well.

2.2 The Transformation

Consider a forest of stars \( \mathcal{F} \). In this section we present a one-parameter family of transformations \( f_{\lambda} : \mathcal{F} \rightarrow \mathcal{E}^n \), each of whose members is a candidate transformation of the forest, \( \mathcal{F} \), onto its “purged” version. After some preliminary definitions in Section 2.2.1, we define the transformation in Section 2.2.2, and then, in Section 2.2.3, we discuss the effect of the various terms in the transformation. In Section 2.2.4 we give a detailed account of the geometrical features for which a knowledge of a bound is required.
Figure 1: A simple tree of stars (top), and two prohibited situations (bottom). On the left, not all the centers of the leaves are inside their respective parent. On the right, two leaves intersect each other, so that the stars intersection arrangement is not a tree.
2.2 The Transformation

2.2.1 Some Preliminary Definitions

First let us introduce some notation. According to Definition 1, an obstacle with corners \( \Omega_n \), is a Boolean combination of analytic manifolds with boundary \( \Omega_{ij} \) for \( j \in \{1, \ldots, L_i\} \). Denote by \( \beta_{ij} \) an "obstacle function" for \( \Omega_{ij} \). That is, let \( \beta_{ij} : E^n \to \mathbb{R} \) be an analytic map for which zero is a regular value, describing \( \Omega_{ij} \) in the form

\[
\Omega_{ij} = \left\{ q \in E^n : \beta_{ij}(q) < 0 \right\} \quad \text{and} \quad \partial \Omega_{ij} = \left\{ q \in E^n : \beta_{ij}(q) = 0 \right\}.
\]

For example, think of the \( \Omega_{ij} \)'s as planar or quadratic polynomial inequalities in \( E^n \). Also denote by \( P_{ij} \) the \( (n-1) \)-dimensional "patch" contributed by \( \Omega_{ij} \) to \( \partial \Omega_n \), that is, \( P_{ij} = \partial \Omega_{ij} \cap \partial \Omega_n \).

Consider a leaf obstacle \( \Omega_i \) for some \( i \in \mathcal{L} \). By construction, \( \Omega_i \) is “connected” to its parent, \( \Omega_{p(i)} \), via a unique patch denoted by \( P_i \). By construction, \( P_i \) is contained in the zero level set of a known polynomial,

\[
\pi_i(q) \triangleq \begin{cases} 
(q - p_i) \cdot v_i - c_i & \text{if } P_i \text{ is planar} \\
\pm \left[ (q - p_i)^T Q_i(q - p_i) - 1 \right] & \text{if } P_i \text{ is quadratic,}
\end{cases}
\]

where \( v_i \) is a non-zero vector, and \( Q_i \) is a positive definite (symmetric) constant matrix. The sign is chosen to satisfy the condition that \( D^2 \pi_i > 0 \) corresponds to a convex patch with respect to the obstacle, and \( D^2 \pi_i < 0 \) to a concave one.

We do not treat the whole list of quadratic shapes. Augmenting the construction presented below with other quadratic shapes — for instance, cylinders and cones — requires more tedious algebra but resembles the ellipsoidal and planar shapes we discuss here.

The following function plays a prominent role in the construction of the transformation.

**Definition 4** The \( i \)th discriminant function, \( \lambda_i \), is the real valued function defined on \( E^n \) by

\[
\lambda_i(q) \triangleq \begin{cases} 
(q - q_i)^T Q_i(q - q_i) - 1 & \text{if } P_i \text{ is planar} \\
(q - q_i) \cdot v_i & \text{if } P_i \text{ is quadratic,}
\end{cases}
\]

Consider the center of the \( i \)th star, \( q_i \). Let \( C_{n_i} \) be the subset of \( E^n \) consisting of all lines through \( q_i \) which intersect \( \pi_i^{-1}(0) \) — the quadratic or planar surface in \( E^n \) containing the patch \( P_i \). In the case of a quadratic patch with \( q_i \) outside the \( (n-1) \)-ellipse containing \( P_i \), \( C_{n_i} \) is a cone with vertex at \( q_i \) boundary tangent to the ellipse \( \pi_i^{-1}(0) \); otherwise \( C_{n_i} \) is \( E^n \) [14, Lemma 2.2].

2.2.2 The Definition of the Transformation

We are now ready to describe the transformation proper. We begin with the complete recipe, and then give its constituent terms: the star-set deforming factors, \( \nu_i \); and the switches, \( \sigma_i \).

Let \( \hat{F} \) denote some neighborhood about the free space \( F \) in \( E^n \). As a notational aid, in certain expressions involving both scalar and vector-valued functions defined on \( E^n \), we will write \( \hat{F} \) to emphasize that \( x \) is vector valued.
Definition 5 Let $F$ be a simple forest of stars. The forest “purging” transformation, $f_\lambda$, is a member of the one-parameter family of continuous maps from $\hat{F} \subset F^m$ into $F^m$, defined by
\[
\tilde{f}_\lambda(q) \triangleq \sum_{j \in \mathcal{L}} \sigma_j(q, \lambda) \tilde{f}_j(q) + \sigma_d(q, \lambda) \bar{q},
\]
where the “ray scaling” maps $\tilde{f}_j$ are defined by,
\[
\tilde{f}_j(q) \triangleq \nu_j(q)(\bar{q} - \bar{q}_j) + \bar{q}_j,
\]
such that $\nu_j$ is the $j^{th}$ star-set deforming factor (Def. 6 below), $\sigma_j$ is the $j^{th}$ switch (Def. 7 below), and $\sigma_d$ is defined by
\[
\sigma_d \triangleq 1 - \sum_{j=0}^{M} \sigma_j.
\]
The forest “modifying” transformation, $h_{\lambda_1, \ldots, \lambda_d}$, is defined by,
\[
h_{\lambda_d, \ldots, \lambda_1}(q) \triangleq f_{\lambda_d} \circ \cdots \circ f_{\lambda_1},
\]
where $d$ is the depth of the deepest tree in $F$, and $f_{\lambda_j}$ for $j \in \{1, \ldots, d\}$ are purging transformations.

Definition 6 The star-set deforming factors, $\nu_i$ for $i \in \mathcal{L}$, are the real valued functions defined on $F$ by
\[
\nu_i(q) \triangleq \begin{cases} 
-\pi_i(q) + \kappa_i(q) & P_i \text{ planar} \\
\delta_i(q) + \kappa_i(q) & P_i \text{ quadratic}, 
\end{cases}
\]
where $\delta_i$ is the $i^{th}$ discriminant function (Def. 4), and $\kappa_i$ is a “correcting term”, defined by
\[
\kappa_i(q) \triangleq \begin{cases} 
2\frac{(\beta_i(q) - \bar{E}_i)}{E_i} \tilde{\kappa}_i(q) \|q - q_i\|^2 & P_i \text{ planar} \\
\left(\frac{\beta_i(q)}{E_i}\right)^2 \left(\tilde{\kappa}_i(q)\right)^2 \|q - q_i\|^2 \|Q_i(q - q_i)\|^2 \|p_i - q_i\|^2 & P_i \text{ quadratic}, 
\end{cases}
\]
where $\tilde{\kappa}_i$ is defined by
\[
\tilde{\kappa}_i(q) \triangleq \beta_{p_i}(q) + (\beta_i(q) - 2E_i) + \sqrt{\beta_i^2(q) + (\beta_i(q) - 2E_i)^2},
\]
and $E_i$ is a geometrical constant defined in Sec. 2.2A below.

Note that $\nu_i$ is defined in terms of the $i^{th}$ obstacle function, $\beta_i$, its parent obstacle function, $\beta_{p_i}$, the polynomial corresponding to the patch “connecting” $C_i$ to its parent, $\pi_i$, and the center of the $i^{th}$ star, $q_i$. First let us verify that $\nu_i$ is well defined. By construction $q_i$ is outside the free space $F$, therefore,
\[
(q - q_i)^T Q_i(q - q_i) > 0 \quad \text{for all } q \in F.
\]
2.2 The Transformation

The "correcting term", $\kappa_i$, is a non-negative function guaranteeing that $\delta_i + \kappa_i > 0$ on $F$, as we show in Lemma B.1 in the Appendix. It follows from the last two facts that $\nu_i$ is well defined on $F$.

Let us sketch the geometrical intuition behind Definition 6 for the quadratic case. Consider

$$\hat{\nu}_i(q) \Delta \nu_i(q)|_{\kappa_i=0} = \frac{(y - y_i)^T Q_i (y - y_i)}{(y - y_i)^T Q_i (y - y_i)}.$$  \hspace{1cm} (9)

It is easy to verify that $\hat{\nu}_i$ is a solution to the following problem. Given a point $q \in E^n$, find a scalar $\hat{\nu}_i$ such that the line $r_q(\hat{\nu}_i) = \hat{\nu}_i (q - y_i) + y_i$ intersects the ellipse $\pi_i^{-1}(0)$ at $q$. This translates to the quadratic equation in $\hat{\nu}_i$,

$$(r_q(\hat{\nu}_i) - p_i)^T Q_i (r_q(\hat{\nu}_i) - p_i) = 1.$$ \hspace{1cm} (10)

whose solutions are given in eq. (9) above. We show in [14] that the $i^{th}$ discriminant function, $\delta_i$, appearing in (9), describes $CN_i$ the cone with vertex at $y_i$ tangent to the ellipse $\pi_i^{-1}(0)$. Thus the solutions (9) become complex for points $q$ outside the cone $CN_i$. In order to stay in the realm of real-valued functions, we modify $\hat{\nu}_i$ by adding the "correcting term" $\kappa_i$: we show in Lemma B.1 in the Appendix that indeed $\delta_i + \kappa_i > 0$ on $F$. The function $\hat{\kappa}_i$ appearing in $\kappa_i$ corresponds to a new "obstacle", obtained by intersecting a "$2E_i$-thickened" version of $C_i$ with $O_{P(i)}$, that is,

$$\{q \in E^n : \hat{\kappa}_i(q) < 0\} = \{q \in E^n : \beta_i(q) < 2E_i\} \cap O_{P(i)}.$$

The role of $\hat{\kappa}_i$ is to guarantee that $\nu_i = \hat{\nu}_i$, on the portion of the parent’s boundary contained in a $2E_i$-neighborhood about $\partial C_i$ in $F$, as we show in Lemma B.1 in the Appendix.

Denote by $\gamma_d$ the (Euclidean) distance to the destination point,

$$\gamma_d(q) \Delta \|q - q_d\|^2.$$ \hspace{1cm} (11)

Definition 7 The switches, $\sigma_i$, for $i \in L$, are the real valued functions defined on $F$ by

$$\sigma_i(q, \lambda) \Delta \frac{x}{x + \lambda} o \frac{\beta_i \hat{\beta}_i}{\beta_i} = \frac{\beta_i \hat{\beta}_i}{\beta_i \hat{\beta}_i + \lambda \beta_i}.$$ \hspace{1cm} (12)

where $\lambda$ is a positive constant, $\beta_i$ is the "omitted product",

$$\hat{\beta}_i \Delta \gamma_d \left( \prod_{k \in I_{-\{i,P(i)\}}} \beta_k \right) \left( \prod_{k \in L_{-\{i\}}} \beta_k \right),$$

and $\hat{\beta}_i$ is defined by

$$\hat{\nu}_i(q) \Delta \beta_{P(i)}(q) + (2E_i - \beta_i(q)) + \sqrt{\beta_{P(i)}^2(q) + (2E_i - \beta_i)^2(q)}.$$

where $E_i$ is a geometrical constant defined in Sec. 2.2.4 below.
The function \( \hat{\beta}_i \) corresponds to a new "obstacle", obtained by excluding from the parent of \( O_i \), \( O_{p(i)} \), a "2\( d \)-thickened" version of \( O_i \), that is,

\[
\{ q \in E^n : \hat{\beta}_i(q) < 0 \} = O_{p(i)} - \{ q \in E^n : \beta_i(q) \leq 2d \}.
\]

Since \( O_{p(i)} \) is one of the "holes" punctured in \( F \), and since \( \hat{\beta}_i \) is negative only on some subset of \( O_{p(i)} \), we have that \( \hat{\beta}_i \geq 0 \) on \( F \). Moreover, \( \hat{\beta}_i \) is strictly positive on the interior of the region excluded from \( O_{p(i)} \). In particular, \( \hat{\beta}_i > 0 \) on the portion of the parent's boundary inside this region. Since the various obstacle functions, \( \beta_i \) for \( i \in I \), are non-negative on \( F \) and positive away from the \( i^{th} \) star boundary, \( \partial O_i \), and since by construction the various trees in \( F \) are as stars in the same tree which are not a "parent-son" pair are disjoint, the switches are well-defined. In fact, the switches are analytic whenever the obstacle functions are analytic. The \( i^{th} \) switch, \( \sigma_i \), maps the interior of the free space to the open interval \((0, 1)\), attains a uniform value of 1 on \( \partial O_i \), and vanishes on any other obstacle boundary except on a portion of its parent's boundary, \( \partial O_{p(i)} \), contained in a \( 2d \)-neighborhood about \( \partial O_i \) in \( F \), on which it varies smoothly between 0 and 1. In the transformation scheme, sufficiently close to the \( i^{th} \) leaf's boundary, the switches provide a means by which the transformation problem is reduced to the simpler problem of mapping one star-shaped leaf onto a portion of its parent's boundary.

### 2.2.3 Discussion of the Transformation

Consider a simple forest of stars \( F \). By construction, each of its \( M+1 \) tree-like obstacles is comprised of several intersecting stars. In addition, \( F \) has a labeling rendering it as simple (Sec. 2.1). Given such an \( F \) with a labeling, we define its purged version, \( \hat{F} \), by

\[
\hat{F} \triangleq F \cup \bigcup_{i \in \mathcal{L}} \left( O_i - O_{p(i)} \right).
\]

That is, the space \( F \) with the leaves in trees consisting of more than one star filled-in and "added back" to \( F \) (by construction, \( O_i \) intersects only its parent, and \( F \) is Euclidean \( n \)-space "punctured" by the various obstacles). In the next section we will prove that for suitable values of the parameter \( \lambda \), the purging transformation maps the forest \( F \) onto its "purged" version, \( \hat{F} \). By composing \( d \) such transformations, \( F \) is mapped onto a "model" star world, on which we already know how to construct a navigation function [16]. In the rest of this section we discuss the geometric intuition behind the construction.

First consider the "destination switch", \( \sigma_d \). Since \( \gamma_d(q) = \| q - q_d \| \) appears as a factor in all the switches, we have that \( f_\lambda(q_d) = q_d \). This is a necessary condition for our "pullback" method composing a navigation function on the purged forest with the purging transformation to work, since the navigation function on the purged forest has a unique minimum at \( q_d \). Intuitively, away from the leaves' boundaries, \( \partial O_i \) for \( i \in \mathcal{L} \), the purging transformation "looks like" the identity map,

\[
f_\lambda(q) \cong \sigma_d(q) \text{id}(q) \cong q,
\]

provided that the parameter \( \lambda \) is sufficiently large, as will be made precise later.
2.2 The Transformation

On the $i^{th}$ leaf's boundary, $\partial \Omega_i$, we have that

$$\sigma_i = 1 \quad \text{and} \quad \sigma_j = 0 \quad \text{for} \quad j \in \mathcal{L} - \{i\},$$

and as a consequence,

$$\tilde{f}_\lambda | \partial \Omega_i = \tilde{f}_i(q) = \nu_i(q)(\tilde{q} - \tilde{q}_i) + \tilde{q}_i,$$

the "ray scaling" map, mapping the $i^{th}$ leaf's boundary onto a portion of its parent's boundary. On the boundary of the $i^{th}$ leaf's parent, $\partial \Omega_{p(i)}$, there are two cases to consider. First, in we have that $\tilde{\beta}_i \geq 0$, and as a consequence,

$$\sigma_i \geq 0 \quad \text{and} \quad \sigma_j = 0 \quad \text{for} \quad j \in \mathcal{L} - \{i\},$$

which implies that

$$\tilde{f}_\lambda | (\partial \Omega_{p(i)} \cap S_i(2E_i)) = \sigma_i [\nu_i(\tilde{q} - \tilde{q}_i) + \tilde{q}_i] + (1 - \sigma_i)\tilde{q}.$$

The geometric constant $E_i$ is defined in Sec. 2.2.4 below as to guarantee that the portion of $\partial \Omega_{p(i)}$ contained in the $2E_i$-neighborhood about $\Omega_i$ in $\mathcal{F}$, $\partial \Omega_{p(i)} \cap S_i(2E_i)$, is contained in the patch $\mathcal{P}_i$ connecting $\Omega_i$ with its parent. Moreover, we show in Lemma B.1 that, by construction,

$$\nu_i | (\mathcal{P}_i \cap S_i(2E_i)) = 1,$$

Substituting for $\nu_i$ yields,

$$\tilde{f}_\lambda | (\partial \Omega_{p(i)} \cap S_i(2E_i)) = \sigma_i [(q - q_i) + q_i] + (1 - \sigma_i)\tilde{q} = \tilde{q}.$$

Consider now the complementary portion of the parent's boundary. It is easy to verify that $\beta_i$ vanishes on this region, which implies that $\sigma_i = 0$ as well, and as a consequence that $f_\lambda$ is the identity map. Similarly, $f_\lambda$ becomes the identity map on the boundary of all the star obstacles not considered so far i.e., any obstacle which is not a leaf nor a parent of a leaf. We conclude that

$f_\lambda$ becomes the "ray-scaling" map, $f_i$, on the $i^{th}$ leaf's boundary, and is the identity map on the boundary of any non-leaf obstacle.

There is no problem with the continuity of $f_\lambda$ on $\partial \mathcal{F}$, since by construction the $i^{th}$ "ray-scaling" map, $f_i$, becomes the identity map at the "corner points" — $\partial \Omega_i \cap \partial \Omega_{p(i)}$.

2.2.4 The Geometrical Information Required

The following definition details the various geometrical features required in the construction of the transformation and later in the proof of its correctness. We assume that these features are available as a part of the data, since they are all derived from the known obstacle functions. In practice, the derivation of these constants from the "obstacle functions" might be computationally intensive. However, all that is required are bounds, not the exact values.
Define the set $S_i(\epsilon)$ by
\[
S_i(\epsilon) \triangleq \{ q \in \mathcal{F} : 0 \leq \beta_i(q) \leq \epsilon \},
\]
describing an $\epsilon$-neighborhood about $\partial C_i$, the boundary of the $i^{th}$ obstacle, in $\mathcal{F}$. Recall that the set of "sharp corners", $C_i \subset \partial C_i$, is thin (nowhere dense) in $\partial C_i$, and that the obstacle function $\beta_i$ is merely continuous on this set.

**Definition 8** The "half gaps", $E_{i1}$, are the positive constants satisfying
\[
S_i(2E_{i1}) \cap S_j(2E_{j1}) = \emptyset \quad \text{for all } i, j \in \mathcal{I} \text{ such that } i \neq j, \ i \neq p(j), \ \text{and } p(i) \neq j;
\]
and
\[
\gamma_d^{-1}[0, E_d] \cap S_i(2E_{i1}) = \emptyset \quad \text{for all } i \in \mathcal{I},
\]
where $\gamma_d = \| q - q_d \|^2$ is the distance to the destination point, and $E_d$ is a positive constant.

The "star-shaped collars", $(\Delta_i, E_{i2})$, are the pairs of positive constants satisfying
\[
\nabla \beta_i \cdot (q - q_i) \geq \Delta_i \quad \text{for all } q \in S_i(E_{i2}) - C_i, \ \text{and for all } i \in \mathcal{L}.
\]

The leaves' "conical neighborhoods", $E_{i3}$, are the positive constants satisfying,
\[
\inf_{\mathcal{F} - T_i} \{ \beta_i \} \geq 3E_{i3} \quad \text{for all } i \in \mathcal{L},
\]
where $T_i$ is the subset of $\mathcal{F}$ defined by
\[
\{ \pi_i(q) \geq 0 \}
\]
for all $q \in \mathcal{F}$, $\pi_i$ planar or quadratic convex with $\pi_i(q_i) < 0$
\[
\{ \delta_i(q) \geq 0, \pi_i(q) \geq \beta_i(q), \nabla \pi_i(q) \cdot (q - q_i) \geq 0, \nabla \pi_i(q) \cdot (q - q_i) \leq 0 \}
\]
for all $q \in \mathcal{F}$, $\pi_i$ quadratic convex with $\pi_i(q_i) \geq 0$
\[
\{ \delta_i(q) \geq 0, \{ \pi_i(q) \geq 0 \text{ or } \nabla \pi_i(q) \cdot (q - q_i) \leq 0 \}, \nabla \pi_i(q) \cdot (q - q_i) \geq 0 \}
\]
for all $q \in \mathcal{F}$, $\pi_i$ quadratic concave.

where convex means $D^2 \pi_i > 0$ and concave $D^2 \pi_i < 0$.

The leaves' "patch neighborhoods", $E_{i4}$, are the positive constants satisfying
\[
\min_{C \cap \mathcal{P}, \mathcal{P}} \{ \beta_i \} \geq 3E_{i4} \quad \text{for all } i \in \mathcal{L},
\]
where $\mathcal{P}$ is the subset of $\mathcal{P}$ defined by

We define $E_i$ by
\[
E_i \triangleq \min \{ E_{i1}, E_{i2}, E_{i3}, E_{i4} \} \quad \text{for all } i \in \mathcal{L},
\]
and
\[
E_i \triangleq E_{i1} \quad \text{for all } i \in \mathcal{I} - \mathcal{L}.
\]

Finally, the leaves' "corner collars", $E_{p(i)}$, are the of positive constants satisfying
\[
\nabla \beta_{p(i)} \cdot (q - q_i) \geq 0 \quad \text{for all } q \in T_i(E_i) \cap S_{p(i)}(E_{p(i)}) - C_{p(i)}, \ \text{and for all } i \in \mathcal{L},
\]
where $S_{p(i)}(E_{p(i)}) = \{ q \in \mathcal{F} : 0 \leq \beta_{p(i)} \leq E_{p(i)} \}$. 

2.2 The Transformation

First consider the "half gaps" $E_{ii}$: the "$2E_{ii}$-thickened" star boundaries still do not intersect (except for "parent-son" pairs), nor do they overlap the destination. By construction, the destination point $q_d$ is in the interior of $\mathcal{F}$, and the closures of the various stars in $\mathcal{F}$ are disjoint (except "parent-son" pairs); thus there is no problem with the definition of $E_{ii}$ and $E_d$. We will use the "half gaps" in the proof that the Jacobian of the transformation is non-singular.

As for the other geometrical features, we refer the reader to [1-4, Sec. 2], in which we provide a detailed discussion of these parameters, their importance in the construction, and, later, in the proof of its correctness.
3 Correctness of the Construction

In Section 3.1 we characterize a diffeomorphism in terms of its Jacobian and its behavior on the boundary components. Then we show that the purging transformation of the previous section, \( f_\lambda \), satisfies these conditions provided it has a non-singular Jacobian away from the “sharp corners”, \( \mathcal{C} \). Finally, in Section 3.2 we prove that the Jacobian of \( f_\lambda \) is indeed non-singular on \( \mathcal{F} - \mathcal{C} \).

3.1 \( f_\lambda \) is an Analytic Diffeomorphism if Its Jacobian is Non-Singular

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be \( n \)-dimensional subsets of \( \mathbb{E}^n \), which are compact connected topological manifolds with boundary, having \( M + 1 \) disjoint boundary components. The following Proposition, used in Theorem 1 below, describes a sufficient condition for a continuous map \( f : \mathcal{X} \to \mathbb{E}^n \) to be a homeomorphism onto a given space \( \mathcal{Y} \subset \mathbb{E}^n \). The notation \( f \in C^q[\mathcal{X}, \mathcal{Y}] \) means that \( \mathcal{X} \) and \( \mathcal{Y} \) have open neighborhoods in \( \mathbb{E}^n \), \( \mathcal{X}' \) and \( \mathcal{Y}' \), such that \( f \in C^q[\mathcal{X}', \mathcal{Y}'] \). We denote the \( j^{th} \) boundary component of a space, \( \mathcal{X} \) say, by \( \partial_j \mathcal{X} \), and say that a map is a bijection if it is one-to-one and onto.

**Proposition 3.1 ([16], Proposition 3.1)** Let \( \mathcal{C} \subset \partial \mathcal{X} \) be closed and thin (nowhere dense) in \( \partial \mathcal{X} \). A continuous map \( f : \mathcal{X} \to \mathbb{E}^n \) such that \( f \in C^q[\mathcal{X} - \mathcal{C}, \mathbb{E}^n] \) for \( q \geq 1 \), is a homeomorphism onto \( \mathcal{Y} \) if

1. \( f \) has a non-singular Jacobian on \( \mathcal{X} - \mathcal{C} \);
2. \( f|_{\partial_j \mathcal{X}} \) is a bijection onto \( \partial_j \mathcal{Y} \) for \( j \in \{0, \ldots, M\} \).

The following Theorem constitutes the central contribution of this report. Given a simple forest of stars, recall our definition of its purged version, \( \hat{\mathcal{F}} \).

\[
\hat{\mathcal{F}} = \mathcal{F} \cup \bigcup_{i \in \mathcal{L}} (\mathcal{O}_i - \mathcal{O}_{p(i)})
\]

that is, the space \( \mathcal{F} \) with the leaves in trees consisting of more than one star filled-in and “re-attached” to \( \mathcal{F} \). In particular, \( \hat{\mathcal{F}} \) has the same number of boundary components as \( \mathcal{F} \).

**Theorem 1** For any simple forest of stars, \( \mathcal{F} \), there exists a positive constant, \( \lambda \), such that if \( \lambda \geq \Lambda \), then the “purging” transformation (Def. 5),

\[
f_\lambda : \mathcal{F} \to \mathbb{E}^n,
\]

has the following properties

1. \( f_\lambda|_{\mathcal{F}} \) is a homeomorphism from \( \mathcal{F} \) onto its purged version \( \hat{\mathcal{F}} \);
2. \( f_\lambda |(\mathcal{F} - \mathcal{C}) \) is an analytic diffeomorphism, where \( \mathcal{C} \subset \partial \mathcal{F} \) is thin in \( \partial \mathcal{F} \), corresponding to "sharp corners";

3. the destination point, \( q_d \), is a fixed point of \( f_\lambda \): \( f_\lambda(q_d) = q_d \);

4. the Jacobian of \( f_\lambda \) is bounded on \( \mathcal{F} - \mathcal{C} \), provided that the gradient of the "obstacle functions", \( \nabla \beta_i \), for \( i \in I \), is bounded on \( \mathcal{F} - \mathcal{C} \).

The proof is given in Appendix B.2.

The set \( \mathcal{C} \) in the Theorem is comprised of three sets: \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_n \cup \mathcal{C}_m \). The first,

\[
\mathcal{C}_1 \triangleq \bigcup_{i \in I} \mathcal{C}_i,
\]

where \( \mathcal{C}_i \subset \partial \mathcal{O}_i \), appeared in the definition of the \( i^{th} \) "obstacle function", \( \beta_i \), as the set of "sharp corners" in \( \partial \mathcal{O}_i \). The second,

\[
\mathcal{C}_n \triangleq \bigcup_{i \in \mathcal{L}} \left( \partial \mathcal{O}_i \cap \partial \mathcal{O}_{p(i)} \right),
\]

is the set of intersection points between the leaves' boundary and their respective parents' boundary\(^2\). The third,

\[
\mathcal{C}_m \triangleq \bigcup_{i \in \mathcal{L}} \left( \beta_i^{-1}[0, 2E_i] \cap \partial \mathcal{O}_{p(i)} \right),
\]

arises in our construction of the switches and the deforming factors, \( \sigma_i \) and \( \nu_i \) for \( i \in \mathcal{L} \). For example, in \( \sigma_i \) there is the function \( \beta_i \) which is merely continuous on \( \mathcal{F} \). The effect of this \( \beta_i \) in \( \sigma_i \) is to render \( \sigma_i \) as unity on \( \partial \mathcal{O}_i \), between zero and one on \( \partial \mathcal{O}_{p(i)} \cap \beta_i^{-1}[0, 2E_i] \), and as zero on the portion of \( \partial \mathcal{O}_{p(i)} \) which is "2\( E_i \) away" from \( \partial \mathcal{O}_i \). Note that \( \mathcal{C} \), being a finite union of thin sets in \( \partial \mathcal{F} \), is thin in \( \partial \mathcal{F} \) as well.

3.2 The Jacobian of \( f_\lambda \) is Non-Singular Away from the "Sharp Corners"

In this section we compute a lower bound on the parameter \( \lambda \), guaranteeing that the Jacobian of \( f_\lambda \) is non-singular on \( \mathcal{F} - \mathcal{C} \). The technical material establishing this result consists of rather tedious calculations that we have relegated to Appendix B.2. It seems worth pausing here for the intuitive motivation.

According to its definition (Def. 5),

\[
\tilde{f}_\lambda(q) = \sum_{i \in \mathcal{L}} \sigma_i(q, \lambda) \tilde{f}_i(q) + \sigma_d(q, \lambda)q_d.
\]

The effect of \( \lambda \) becomes clear if one inspects the role of the switches, \( \sigma_i \), for \( i \in \mathcal{L} \). By construction, at any point \( q \in \mathcal{F} \),

\[
0 \leq \sigma_i(q, \lambda) \leq 1 \quad \text{for} \quad i \in \mathcal{L}.
\]

\(^2\)For \( \mathcal{C}_n \) to be thin in \( \partial \mathcal{F} \) we have to exclude the possibility that \( \partial \mathcal{O}_i \) and \( \partial \mathcal{O}_{p(i)} \) share a common patch.
3 CORRECTNESS OF THE CONSTRUCTION

We shall see that for any fixed $\epsilon > 0$, "away" from the $i$th leaf's boundary i.e., at points $q \in F$ for which $\beta_i(q) > \epsilon$, $\sigma_i$ can be made arbitrarily small by increasing $\lambda$. Thus, if the value of $\lambda$ is sufficiently large, away from the leaves' boundary the destination switch, $\sigma_d = 1 - \sum_{i \in \mathcal{L}_i} \sigma_i$, is approximately unity, and $f_\lambda$ essentially looks like a perturbed identity mapping,

$$f_\lambda(q) = \text{id}(q) + \epsilon f(\sigma_0, \ldots, \sigma_{L})$$

where $f$ depends linearly on the switches $\sigma_i$. We shall see as well that at any such point, $\sigma_i$ can be made arbitrarily "flat" - that is, $\|\nabla \sigma_i(q, \lambda)\|$ can be made arbitrarily small by increasing $\lambda$. Since $Df_\lambda$ depends linearly on $\sigma_i$ and $\nabla \sigma_i$ for $i \in \mathcal{L}$, it follows that away from the leaves' boundary, $f_\lambda$ is essentially the identity transformation of $E^n$, and its Jacobian is dominated by the identity matrix.

Thus the real difficulty is the "sharpening" of the switches in a neighborhood about the leaves' boundary as $\lambda$ increases (since $\sigma_i$ is unity on $\partial \mathcal{O}_i \cap F$, no matter how close to zero we make it "away" from $\partial \mathcal{O}_i \cap F$). Consider an $\epsilon$-neighborhood about $\partial \mathcal{O}_i$ in $F$, for $\epsilon$ sufficiently small this neighborhood becomes disjoint from $\partial \mathcal{O}_i$ for $i \neq i$, except the parent's boundary, $\partial \mathcal{O}_{p(i)}$. We presume to ignore the effect of the other obstacles in this neighborhood and regard $f_\lambda$ and its Jacobian as perturbed versions of $f_\lambda|\partial \mathcal{O}_i$ and $Df_\lambda|\partial \mathcal{O}_i$. On $\partial \mathcal{O}_i$ itself, $f_\lambda$ becomes the "ray scaling" map,

$$\tilde{f}_\lambda|\partial \mathcal{O}_i = \tilde{f}_i(q) = \hat{\nu}_i(q)(\bar{q} - \bar{q}_i) + \bar{q}_i$$

where $\hat{\nu}_i$ is bounded away from zero in $\partial \mathcal{O}_i \cap F$. By construction, for each $q \in P_i$, we have that $f_i^{-1}(f_i(q))$ contains the whole line $r_q$ and as a consequence,

$$\frac{d}{ds}(f_i \circ r_q)(s) = [Df_i(r_q(s))(q - q_i) = \left[\hat{\nu}_i(q)I - (q - q_i)\nabla \hat{\nu}_i(q)\right] (q - q_i) = 0,$$

as we discuss in [14, Lemma C.9]. On the other hand, the restriction of $Df_i$ to the $(n-1)$-dimensional subspace orthogonal to $\nabla \nu_i$ is the matrix $\hat{\nu}_i(q)I$, where $\nu_i$ is bounded away from zero in $\partial \mathcal{O}_i$, [14, Lemma C.5]. Thus, the behavior of $Df_\lambda$ on the subspace spanned by $(q - q_i)$ requires special attention. By construction, $\mathcal{O}_i$ is strictly star shaped and therefore satisfies the "star-shapeness" condition (Def. 9 in the Appendix),

$$\nabla \lambda(q) \cdot (q - q_i) \geq 2\Delta_i > 0 \text{ for all } q \in \partial \mathcal{O}_i,$$  \hspace{1cm} (15)

where $q_i$ is the center of $\mathcal{O}_i$. Since the $i$th switch, $\sigma_i$, is unity on $\partial \mathcal{O}_i = \beta_i^{-1}(0)$ and, depending on the value of the parameter $\lambda$, drops toward zero away from $\partial \mathcal{O}_i$, we have that

$$\nabla \sigma_i|\partial \mathcal{O}_i = -\sigma(q)\nabla \hat{\nu}_i,$$

for some positive real valued function $\sigma(q)$. Using (16),

$$\|\nabla \sigma_i|\partial \mathcal{O}_i \cdot (q - q_i) \| \geq 2\sigma(q)\Delta_i,$$
and we will exploit this property to demonstrate that $Df_\lambda$ is non-singular along $(q - q_i)$. We shall see that by further restricting $\lambda$, this property can be extended to a neighborhood about $\partial \Omega_i$ in $\mathcal{F}$. Thus our plan will be to show that for each $i \in \mathcal{L}$ $f_\lambda$ has a non-singular Jacobian in an $\epsilon$-neighborhood about $\partial \Omega_i$ in $\mathcal{F}$ for $\epsilon > 0$ sufficiently small, and then to use this $\epsilon_i$ to show that by further constraining the parameter $\lambda$, the Jacobian of $f_\lambda$ is non-singular "$\epsilon$ away" from the leaves' boundary, as we have discussed above. We may now proceed with the formal proof, for which we refer the reader to Appendix B.3.
4 Counting the Floating Point Operations

The computation involved has two parts. First, when presented with the geometrical data (Sections 1.3 and 2.2.4), describing the forest \( F \), we construct a navigation function on \( F \) in the form,

\[
\varphi = \varphi_s \circ (f_{\lambda_d} \circ \cdots \circ f_{\lambda_1}),
\]

where \( \varphi_s \) is a navigation function on the corresponding star world, \( F_s \), whose star obstacles are the roots of the various trees in \( F \), and \( f_{\lambda_i} \) for \( i \in \{1, \ldots, d\} \) are successive purging transformations, mapping \( F \) onto \( F_s \). The construction of \( \varphi \) involves the choice of \( d+2 \) parameters for the \( d+1 \) functions comprising \( \varphi \) (\( \varphi_s \) has two parameters). The computational complexity of this part is analysed in Section 4.1. Second, the controller has to compute \( \nabla \varphi \),

\[
\nabla \varphi = \nabla (\varphi_s \circ f_i \circ \cdots \circ f_1) = [Df_1]^T [Df_2(f_1)]^T \cdots [Df_d(f_{d-1} \circ \cdots \circ f_1)]^T \nabla \varphi_s(f_d \circ \cdots \circ f_1). \tag{17}
\]

where \( f_i \) is a shorthand notation for \( f_{\lambda_i} \). We analyze the computational complexity of this term in Section 4.2.

4.1 The Computation of the Parameters in \( \varphi \)

The count of the floating point operations is given in terms of \( M \) — the number of tree-like obstacles, \( d \) — the depth of the forest \( F \), and \( n \) — the dimension of the ambient Euclidean space. In addition, there is the computation of the geometrical data. Specifically, for each of the star obstacles in \( F \) we have required the knowledge of a list of constants (Sec. 2.2.4), \( E_i \), \( E_{p(i)} \), and \( \Delta_i \) for \( i \in L \); bounds on the “radius” of each star,

\[
\min_F \{\|q - q_i\|\} \quad \text{and} \quad \max_F \{\|q - q_i\|\} \quad \text{for} \quad i \in I; \tag{18}
\]

as well as upper bounds on the obstacle functions and their normed gradient,

\[
\max_F \{\beta_i\} \quad \text{and} \quad \max_F \{\||\nabla \beta_i||\} \quad \text{for} \quad i \in I. \tag{19}
\]

Let us denote the computational cost of these terms by the prefix \( \# \). For instance, the computational cost of \( E_i \) is denoted by \( \#(E_{p(i)}) \). Then the total computational cost of the data terms, for all the \( d \) purging transformations, is bounded from above by

\[
\sum_{i \in I} \left( \#(E_i) \times \#(E_{p(i)}) \times \#(\Delta_i) \right) + \#(\min_F \{\|q - q_i\|\}) + \#(\max_F \{\|q - q_i\|\}) + \#(\max_F \{\beta_i\}) + \#(\max_F \{\||\nabla \beta_i||\}) \tag{20}.
\]

We plan to develop explicit formulas for this data in a future report: since each star obstacle is rendered as a Boolean combination of planar and quadratic polynomial inequalities in \( \mathbb{R}^n \), and since we ask for are bounds, not exact values, such formulas ought to be readily obtainable. In this paper, however, we use these symbols as “place holders” in the total count. All these data terms involve bounds on the maximum or minimum of various continuous functions attained on compact subsets of \( F \). In the simulation studies (Sec. 5 below) we find an aspect of a trade-off.
between the "tightness" of these bounds and the qualitative behavior of the resulting navigation function. As the bounds become more conservative and their computation less intensive the various "valleys" leading to the destination point tend to "hug" the obstacles' boundaries more tightly. This causes numerical implementation problems whose amelioration is the subject of research now in progress.

In addition to the computational cost of the data terms, the total number of floating-point operations required is shown in [14] to be bounded by

\[ 10M^2 n + 15M^2 + 25d |I| + |I| n^3; \]

where \( M \) is the number of tree-like obstacles, \( n \) the dimension of the ambient Euclidean space, \( d \) the depth of the forest, and \( |I| \) the number of star obstacles in the forest \( F \).

### 4.2 The Computation of \( \nabla \varphi \)

Denote the number of floating point operations required to compute the \( i^{th} \) obstacle function and its gradient, \( \beta_i \) and \( \nabla \beta_i \), by \( \#(\beta_i) \) and \( \#(\nabla \beta_i) \). The count of the floating point operations will be given in terms of the parameters \( M, d \) and \( n \), as well as \( \#(\beta_i) \) and \( \#(\nabla \beta_i) \) for \( i \in I \). According to eq. (17) above, the computation of \( \nabla \varphi(q) \) involves the following steps.

1. compute \( p_1 \triangleq f_{\lambda_1}(q), p_2 \triangleq f_{\lambda_2}(p_1), \ldots, p_d \triangleq f_{\lambda_d}(p_{d-1}) \);
2. compute \( \nabla \varphi(p_d) \);
3. compute \( Df_{\lambda_1}(q), Df_{\lambda_2}(p_1), \ldots, Df_{\lambda_d}(p_{d-1}) \);
4. multiply the matrices \( Df_{\lambda_1} \), \( \ldots, Df_{\lambda_d} \) by the vector \( \nabla \varphi_q \).

We show in [14] that that it takes no more than

\[ 5d |I| + 15 |I| n^2 + 5dn^3 + 5d \sum_{j \in I} \left( \# \beta_j + \# \nabla \beta_j \right). \]

operations to compute \( \nabla \varphi(q) \).

The terms \( \# \beta_i \) and \( \# \nabla \beta_i \) are not completely unknown: each of the obstacles is rendered as a Boolean combination of linear and quadratic polynomial inequalities in \( F^n \), and the corresponding obstacle function is constructed according to the "calculus" of implicit representations presented in Appendix A. Suppose that \( \beta_i \) is constructed from \( m \) planar or quadratic inequalities. We show in the same Appendix that the computation of \( \beta_i \), and hence of \( \nabla \beta_i \), can be effectively arranged in a recursive form, so that the computational cost involves the evaluation of square-root functions whose number is linear in \( m \).

In order to appreciate this count, let us consider obstacle functions which are polynomials of degree \( k \) or less. In general, such polynomials consist of \( k \) homogeneous polynomials, each
of which can have no more than \( \binom{n + k - 1}{k} \) terms, therefore,

\[
\#(\beta_i) = k \binom{n + k - 1}{k} \quad \text{and} \quad \#(\nabla \beta_i) = n(k - 2) \binom{n + k - 2}{k - 1}.
\]

Thus, if we relate \( k \) to the “geometric complexity”, and \( n \) to the number of degrees of freedom of the underlying kinematic chain, then, assuming that \( k \geq n \), the computation involved is proportional to \( k^n \), i.e. \textit{exponential} in the number of degrees of freedom and polynomial in the geometric complexity.

It seems intuitively clear (but we have not established rigorously) how this apparent discrepancy resolves. The number of patches required to represent even simple shapes, for instance an \( n \)-cube, grows exponentially with the dimension of the ambient Euclidean space. In general however, a shape is described in terms of a set of polynomials of some high degree \( k \). Clearly, the dependence of the number of linear and quadratic inequalities required to approximate a given higher degree polynomial with some fixed finite precision on the dimension of the ambient space should be investigated. We are not aware of any work on this problem.
5 Simulation Studies

Two numerical examples are provided in this section: a star world with "sharp corners", and a forest of stars.

First, in Figure 2, we show a scene of a bin with various industrial parts. Each of these parts is star shaped with "sharp corners", and satisfies our definition for an obstacle with corners (Def. 1). Namely, each obstacle function is constructed from a Boolean combination of linear and quadratic polynomial inequalities. The resulting space is a planar \( n = 2 \) star world, \( \mathcal{F} \), with nine internal obstacles. The corresponding model sphere world, \( \mathcal{M} \), is also shown. We plot the level lines of a navigation function \( \varphi \) on \( \mathcal{F} \), as well as those of the corresponding navigation function on \( \mathcal{M} \), \( \dot{\varphi} \). The parameter in \( \dot{\varphi} \) is chosen sufficiently high to eliminate spurious local minima in \( \mathcal{M} \), according to the results of [10]. The destination point in both spaces is chosen arbitrarily at \((-1, -0.8)\), and the level lines vary between zero (at the destination point), and one (on all the boundary components). The navigation function on the star world is \( \varphi = \dot{\varphi} \circ h_\lambda \), where \( h_\lambda \) is the star-world to sphere-world transformation. It can be seen that \( h_\lambda \), for an appropriately chosen \( \lambda \), introduces no additional critical points, as guaranteed by the results of [16]. Thus there is a unique minimum at the destination point, and one saddle point near each (internal) star obstacle. As we have shown in [10], one cannot do better than this using smooth vector fields which are transverse to the boundary of \( \mathcal{F} \). In Figure 3 we plot the enlarged image of some interesting regions in \( \mathcal{F} \), demonstrating the "reappearance" of the valleys leading to the destination point.

Next we show a forest of stars resembling a floor plan in a building. There are three internal tree-like obstacles, and the depth of the deepest tree is \( d = 4 \). According to our method, the purging transformation, \( f_{\lambda_1} \), is applied \( d \) times, until a space whose obstacles are the roots of the original trees is obtained. This space is a star world, and we apply our previously constructed star-world to sphere-world transformation, \( h_\lambda \) [16], to obtain the corresponding model sphere world, \( \mathcal{M} \). Thus the sequence of transformations is

\[
\mathcal{F} \xrightarrow{f_{\lambda_1}} \mathcal{F}_1 \xrightarrow{f_{\lambda_2}} \mathcal{F}_2 \xrightarrow{f_{\lambda_3}} \mathcal{F}_3 \xrightarrow{f_{\lambda_4}} \mathcal{F}_4 \xrightarrow{h_\lambda} \mathcal{M}.
\]

We show each of the “intermediate” spaces, as well as the level lines of the navigation function as it is “pulled back” via these spaces.

\[
\dot{\varphi} : \mathcal{M} \to [0, 1], \quad \dot{\varphi} \circ h_\lambda : \mathcal{F}_4 \to [0, 1], \quad \ldots \varphi = \dot{\varphi} \circ h_\lambda \circ f_{\lambda_4} \circ f_{\lambda_3} \circ f_{\lambda_2} \circ f_{\lambda_1} : \mathcal{F} \to [0, 1].
\]

The simulations, while corroborating the theory, reveal certain numerical difficulties. The level lines clearly depict a unique minimum at the destination point. But when the various “valleys” approach the obstacles, they are so close to the boundary that they can be seen again only on the far side of each obstacle, as we portray in Figure 4. We have found from further numerical experimentation that such situations — when the valleys "hug" the boundaries too tightly — lead to gradient vector fields that vary too abruptly to be implemented in a practical setting. The phenomenon becomes even more acute when we increase the parameter values. Of course, the theory requires in general that these values be increased to avoid spurious local minima.
To understand this behavior, consider the purging transformation,

\[ f_{\lambda}(q) = \sum_{i \in \mathcal{L}} \sigma_i(q, \lambda) f_i(q) + (1 - \sum_{i \in \mathcal{L}} \sigma_i(q, \lambda)) \text{id}(q). \]

The effect of increasing the parameter \( \lambda \) is to make the switches, \( \sigma_i \) for \( i \in \mathcal{L} \), vanish more rapidly away from the obstacles' boundary, rendering the transformation essentially the identity mapping. Thus, all the "interesting" features are confined to small neighborhoods about the obstacles' boundaries.

Consider now two sets of plots, displaying the effect of the purging transformations. In the first — Figure 5, we use high parameter values. As in the previous example, the parameter in the sphere-world navigation function, \( \varphi \), is chosen sufficiently high to eliminate spurious local minima in \( \mathcal{M} \) [10], the destination point is is chosen arbitrarily at the origin, and the level lines vary regularly between zero and one. It can plausibly be seen that the transformation of \( \mathcal{F} \) to \( \mathcal{M} \), with valid parameter values, introduces no additional critical points. Thus there is a unique minimum at the destination point, and one saddle point near each (internal) tree-like obstacle.

In the second set — Figure 6, we have experimented with various intuitive numerical remedies for the problem of "disappearing" valleys. Additional parameters introduced in the switches, \( \sigma_i(q, \lambda) \) for \( i \in \mathcal{L} \), make their value close to unity in a neighborhood about the boundaries, and only then decrease to zero. Intuitively, this slows the transition of the purging transformation from the "ray scaling" maps, \( f_i \) for \( i \in \mathcal{L} \), to the identity map, and as a consequence the valleys move from the obstacles' boundary. In Figure 7 we plot the enlarged image of some interesting regions in \( \mathcal{F} \), demonstrating the "reappearance" of the valleys.

It becomes clear from these simulations that the practicability of our construction depends on the development of some control mechanism for the location of the valleys. This is the subject of research now in progress.
Figure 2: Planar star world (bottom), $\mathcal{F}$, and its model sphere world (top), for the case of $M = 9$ internal star-shaped obstacles.
Figure 3: The enlarged image of some regions in Figure 2.
Figure 4: The phenomenon of "disappearing" valleys. The valleys are clearly depicted for low parameter values (top) but disappear as the parameter values increase (bottom).
Figure 5: Planar forest of stars with three internal tree-like obstacles (bottom right), its "purged" versions, and its model sphere world (top left).
Figure 6: Result of experimentation on the space of Figure 5
Figure 7: The enlarged image of some regions in Figure 6.
A "Calculus" for Implicit Representations

In this section we present formulas, introduced by [20], for constructing the implicit representation of a set described by an arbitrary union and intersection of other sets, each of whose implicit representation is already known. If \( m \) sets in \( E^n \) are described by the inequalities

\[
\beta_i(x_1, \ldots, x_n) \leq 0 \quad \text{for} \ i \in \{1, \ldots, m\},
\]

then their intersection is given by \( \psi_\cap(\beta_1, \ldots, \beta_m) \leq 0 \), and their union by \( \psi_\cup(\beta_1, \ldots, \beta_m) \leq 0 \). The functions \( \psi_\cap \) and \( \psi_\cup \) are defined as follows,

\[
\psi_\cap(\beta_1, \ldots, \beta_m) \triangleq \sum_{k=1}^{m} \beta_k + \sum_{k=2}^{m} (-1)^{k} \alpha_k \quad \text{and} \quad \psi_\cup(\beta_1, \ldots, \beta_m) \triangleq \sum_{k=1}^{m} \beta_k + \sum_{k=2}^{m} (-1)^{k+1} \alpha_k.
\]

where

\[
\alpha_1 \triangleq \beta_1 + \beta_2 + \ldots + \beta_m,
\]

\[
\alpha_2 \triangleq \sum_{j=2}^{m} \sqrt{\beta_1^2 + \beta_j^2} + \sum_{j=3}^{m} \sqrt{\beta_2^2 + \beta_j^2} + \ldots + \sqrt{\beta_{m-1}^2 + \beta_m^2},
\]

\[
\alpha_k \triangleq \sum_{j=k}^{m} \sqrt{\beta_1^2 + \beta_2^2 + \ldots + \beta_{k-1}^2 + \beta_j^2} + \ldots + \sqrt{\beta_{m-k+1}^2 + \beta_{m-k+2}^2 + \ldots + \beta_m^2},
\]

\[
\alpha_m \triangleq \sqrt{\beta_1^2 + \beta_2^2 + \ldots + \beta_m^2}.
\]

There are \( \binom{m}{k} \) terms in each \( \alpha_k \). Supposing that each \( \beta_i \) is analytic on \( E^n \), \( \psi_\cap \) and \( \psi_\cup \) are analytic on \( E^n - \mathcal{C} \), where \( \mathcal{C} \) consists of the intersection points of two or more zero level sets of the \( \beta_i \)'s.

**Example:** For \( m = 3 \), we have that \( \psi_\cup = \alpha_1 + \alpha_2 + \alpha_3 \), where

\[
\alpha_1 = \beta_1 + \beta_2 + \beta_3,
\]

\[
\alpha_2 = - \left( \sqrt{\beta_1^2 + \beta_2^2} + \sqrt{\beta_1^2 + \beta_3^2} + \sqrt{\beta_2^2 + \beta_3^2} \right),
\]

and

\[
\alpha_3 = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}.
\]

Thus,

\[
\psi_\cup = \beta_1 + \beta_2 + \beta_3 - \sqrt{\beta_1^2 + \beta_2^2} - \sqrt{\beta_1^2 + \beta_3^2} - \sqrt{\beta_2^2 + \beta_3^2}.
\]

The following Lemma confirms, for \( m = 3 \), that \( \psi_\cap \) faithfully "encodes" the respective set operation.
Lemma A.1 ([20], Theorems 3, 4) Let $S_1, S_2, S_3$ be sets in $E^n$ described by the inequalities,

$$
\beta_1(x_1, \ldots, x_n) \leq 0, \quad \beta_2(x_1, \ldots, x_n) \leq 0, \quad \beta_3(x_1, \ldots, x_n) \leq 0.
$$

We have that

$$
S_1 \cap S_2 \cap S_3 = \{ y \in E^n : \psi_0(\beta_1, \beta_2, \beta_3) \leq 0 \}.
$$

and that

$$
\partial (S_1 \cap S_2 \cap S_3) = \{ y \in E^n : \psi_0(\beta_1, \beta_2, \beta_3) = 0 \}.
$$

An analogous result is given in the same paper for $\psi_{U}$.

We have used these formulas in two places. First, we required in Section 1.2 that the “obstacle functions”, $\beta_i$ for $i \in I$, will be continuous on $E^n$ and analytic on $E^n - C$, where $C \subset \partial F$ is nowhere dense in $\partial F$. In doing so, we presumed that the various obstacles would be described with arbitrary precision by applying the above formulas to a catalog of simple shapes — in our case, half spaces and ellipsoids. The second place was in the construction of the switches and the deforming factors, $\sigma_i$ and $\nu_i$ for $i \in I$, where we have used $\psi_0$. For example, in the $i^{th}$ switch,

$$
\sigma_i = \frac{\tilde{\beta}_i}{\beta_i \tilde{\beta}_i + \lambda \beta_i},
$$

we have that

$$
\tilde{\beta}_i = \psi_{p(i)} + (2E_i - \beta_i) + \sqrt{\alpha_{p(i)}^2 + (2E_i - \beta_i)^2} = \psi_{0}(\beta_{p(i)}, 2E_i - \beta_i).
$$

That is, $\tilde{\beta}_i$ designates the portion of $O_{p(i)}$ which is “$2E_i$ away” from $\partial O_i = \beta_i^{-1}(0)$.

If there are $m$ sets, then each $\alpha_k$ has \( \binom{m}{k} \) terms, so that $\psi_0$ and $\psi_{U}$ have \( \sum_{k=1}^{m} \binom{m}{k} = 2^m - 1 \) terms. That is, the computational complexity of $\psi_0$ and $\psi_{U}$ is exponential in the number of sets $m$, in addition to the computational cost of the $\beta_i$'s.

The situation becomes much better if one is willing to tolerate “nested” square-root functions. Consider $\psi_{U}$. Since union is associative, we can pair the $m$ sets in a binary tree structure. For instance, if $m = 7$, we can construct $\psi_{U}$ as follows.

$$
\psi_{U} = \psi_{U}(\psi_{U}(\psi_{U}(\psi_{U}(\psi_{U}(\psi_{U}(\psi_{U}(\psi_{U}(\beta_1, \beta_2), \psi_{U}(\beta_3, \beta_4)), \psi_{U}(\psi_{U}(\beta_5, \beta_6), \beta_7))))).
$$

Since the number of (internal) nodes in a balanced binary tree with $m$ leaves is $m - 1$, $\psi_{U}$ “invokes” itself roughly $m$ times. Each of these invocations computes a term of the form

$$
\beta_i + \beta_j - \sqrt{\beta_i^2 + \beta_j^2},
$$

so that the computational complexity of $\psi_{U}$ is linear in $m$; in addition to the computational cost of the $\beta_i$'s.
B Some Details

B.1 Details of the Construction

According to Definition 1, an obstacle with corners, $\mathcal{O}_i$, is a Boolean combination of analytic manifolds with boundary $\mathcal{O}_i$ for $j \in \{1, \ldots, L_i\}$. We have denoted by $\beta_{ij}$ an “obstacle function” for $\mathcal{O}_i$. That is, the analytic map $\beta_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$ describing $\mathcal{O}_i$ in the form

$$\mathcal{O}_i = \{ q \in \mathbb{R}^n : \beta_{ij}(q) < 0 \} \quad \text{and} \quad \partial \mathcal{O}_i = \{ q \in \mathbb{R}^n : \beta_{ij}(q) = 0 \}.$$

We have also denoted by $\mathcal{P}_{ij}$ the $(n-1)$-dimensional “patch” contributed by $\mathcal{O}_i$ to $\partial \mathcal{C}_i$, that is, $\mathcal{P}_{ij} = \beta_{ij}^{-1}(0) \cap \partial \mathcal{O}_i$. The following definition is an adjustment of our previous definition of star obstacles (Def. 5 in [16]) to the more general situation of obstacles with corners.

**Definition 9** An obstacle with corners (Def. 1), $\mathcal{O}_i$, is strictly star shaped if there is a point $q_i \in \mathcal{O}_i$ and a positive constant $\Delta_i$, such that

1. the $i^{th}$ obstacle function, $\beta_i$, satisfies $^3$

$$\nabla \beta_i(q) \cdot (q - q_i) \geq 2\Delta_i \quad \text{for all } q \in \partial \mathcal{O}_i - \mathcal{C}_i,$$

where $\mathcal{C}_i \subset \partial \mathcal{O}_i$ are the corner points in $\partial \mathcal{O}_i$;

2. each of the “patches”, $\mathcal{P}_{ij}$ for $j \in \{1, \ldots, L_i\}$, satisfies

$$\nabla \beta_{ij}(q) \cdot (q - q_i) \neq 0 \quad \text{for all } q \in \mathcal{P}_{ij}.$$

If all the obstacles in the free space, $\mathcal{F}$, are strictly star shaped, then $\mathcal{F}$ is called a star world.

Recall that an obstacle with corners (Def. 1) is comprised of $n$-dimensional shapes, $\mathcal{O}_i$, for $j \in \{1, \ldots, L_i\}$, and that for each $\mathcal{O}_i$, we assume the availability of its own “obstacle function”, $\beta_{ij}$ for $j \in \{1, \ldots, L_i\}$; each $\beta_{ij}$ contributes an $(n-1)$-dimensional “patch” $\mathcal{P}_{ij} = \beta_{ij}^{-1}(0) \cap \mathcal{C}_i$.

**Definition 10** A quadratic tree of stars is simple if it has the following properties with respect to some labelling.

1. for each star, $\mathcal{O}_i$, its center point, $q_i$, is contained in the parent star, $\mathcal{O}_{p(i)}$;

2. each star is “connected” to its parent via a unique quadratic patch, $\mathcal{P}_{p(i)}$.

$$\overline{\mathcal{O}_i} \cap \partial \mathcal{O}_{p(i)} \subseteq \mathcal{P}_{p(i)},$$

where $\mathcal{P}_{p(i)}$ is a patch in $\partial \mathcal{O}_{p(i)}$.

$^3$The zeroth obstacle case, describing the outer boundary of $\mathcal{F}$, should be read as $q_0 \in \mathbb{R}^n - \overline{\mathcal{O}}_i$ such that $\nabla \beta_i \cdot (q - q_0) \leq -2\Delta_i$. 


3. the patch $\mathcal{P}_{p(i)}$ is strictly star shaped with respect to $q_i$, that is,

$$\nabla \beta_{p(i)}(q) \cdot (q - q_i) > 0 \quad \text{for all } q \in \mathcal{P}_{p(i)},$$

where $\beta_{p(i)}$ is the "obstacle function" corresponding to $\mathcal{P}_{p(i)}$.

A quadratic forest, $\mathcal{F}$, is simple if each of its trees is simple.

The requirement (23) guarantees the existence of a neighborhood about $\partial \mathcal{O}_i$ in $\mathcal{F}$, whose intersection with $\partial \mathcal{O}_{p(i)}$ is contained in $\mathcal{P}_{p(i)}$. We will use this neighborhood in the construction of the transformation (Def. 8). The requirement (24) can be relaxed to hold in some subset of $\mathcal{P}_{p(i)}$ which is a closed neighborhood about $\overline{C_i} \cap \partial \mathcal{O}_{p(i)}$ in $\partial \mathcal{O}_{p(i)}$. To simplify our notation, denote the patch $\mathcal{P}_{p(i)}$ and its obstacle function $\beta_{p(i)}$ by $\mathcal{P}_i$ and $\pi_i$.

The following Lemma is used in the discussion following the definition of the deforming factors, $\nu_i$ for $i \in \mathcal{L}$ (Sec. 2.2.2).

**Lemma B.1** The "correcting term", $\kappa_i$, possesses the following three properties. First,

$$\delta_i(q) + \kappa_i(q) > 0 \quad \text{for all } q \in \mathcal{F};$$

second,

$$\kappa_i |(\partial \mathcal{O}_{p(i)} \cap \mathcal{S}_i(2E_i)) = 0,$$

and as a consequence,

$$\nu_i |(\mathcal{P}_i \cap \mathcal{S}_i(2E_i)) = 1.$$

Where $\mathcal{P}_i = \partial \mathcal{O}_{p(i)} \cap \pi_i^{-1}(0)$ is the patch "connecting" $\mathcal{O}_i$ with its parent $\mathcal{O}_{p(i)}$.

**Proof:** First let us establish that $\kappa_i(q) \geq 0$ on $\mathcal{F}$. The function $\kappa_i(q)$ in $\mathcal{F}$ is non-negative on $\mathcal{F}$ since $\mathcal{F} \subset F_i^* - \mathcal{O}_{p(i)}$ and $\kappa_i$ is non-negative outside $\mathcal{O}_{p(i)}$. This, together with the positive definiteness of $Q_i$, implies that $\kappa_i(q) \geq 0$ on $\mathcal{F}$. As a consequence, it suffices to consider only those points $q \in \mathcal{F}$ at which $\delta_i(q)$ is non-positive. According to the definition of $E_i$ (Specifically, the "conical neighborhoods"),

$$\beta_i(q) \geq 3E_i \quad \text{for all } q \in \mathcal{F} \text{ such that } \delta_i(q) \leq 0,$$

in particular, $\frac{\beta_i}{E_i} \geq 1$. Furthermore, whenever $\delta_i(q) \leq 0$ we have that $\beta_i(q) - 2E_i \geq E_i$, and as a consequence,

$$\kappa_i = \beta_i(0) + (\beta_i - 2E_i) + \sqrt{\beta_i^2 + (\beta_i - 2E_i)^2} \geq (\beta_i - 2E_i) + |\beta_i - 2E_i| \geq 2E_i.$$

Thus,

$$\frac{\kappa_i(q)}{E_i} \geq 1 \quad \text{for all } q \in \mathcal{F} \text{ such that } \delta_i(q) \leq 0.$$

The last two facts imply that

$$\kappa_i(q) \geq \begin{cases} |(q - q_i)^T Q_i(q - q_i)|[(q - q_i)^T Q_i(q - q_i)] & \text{if } \mathcal{P}_i \text{ is quadratic} \\ 2\|q - q_i\|_i \nu_i & \text{if } \mathcal{P}_i \text{ is planar.} \end{cases} \tag{25}$$
Thus, for a planar patch,

\[ \delta_t(q) + \kappa_t(q) \geq (q - q_i) \cdot v_i + 2\|q - q_i\|\|v_i\| \geq \|q - q_i\|\|v_i\| > 0, \]

since, by construction, \( q_i \) is outside \( \mathcal{F} \); and for a quadratic patch, since the only negative term in \( \delta_t \) can be \( [1 - (p_i - q_i)^T Q_i (p_i - q_i)] \) — describing a situation in which \( q_i, \) the center of \( \mathcal{Q}_i, \) happens to be outside the ellipsoid \( \pi_i^{-1}[-1, 0] \) (or \( \pi_i^{-1}[0, 1] \)) — we have that

\[
\delta_t(q) + \kappa_t(q) \geq [(q - q_i)^T Q_i (p_i - q_i)]^2 + (q - q_i)^T Q_i (q - q_i) [1 - (p_i - q_i)^T Q_i (p_i - q_i)] \\
+ [(q - q_i)^T Q_i (q - q_i)][(p_i - q_i)^T Q_i (p_i - q_i)] \\
= [(q - q_i)^T Q_i (p_i - q_i)]^2 + (q - q_i)^T Q_i (q - q_i) > 0,
\]

since \( q_i \) is outside \( \mathcal{F} \).

Turning to the second property, it follows from the definition of \( E_i \) (specifically, the “patch neighborhoods”) that

\[
\partial \mathcal{Q}_{\beta_i} \bigcap S_i(2E_i) \subset P_i \subset \beta_{\beta_i}^{-1}(0),
\]

which implies that

\[
\kappa_t(q) = (\beta_i - 2E_i) + |\beta_i - 2E_i| = 0,
\]

since \( \beta_i - 2E_i \leq 0 \) in \( S_i(2E_i) \); and as a consequence \( \kappa_t(q) = 0 \).

For the proof that \( \nu_t(q) = 1 \) at any point \( q \) in \( P_i \cap S_i(2E_i) \), we refer the reader to [11, Lemma C.5].

\[ \Box \]
B.2 A Proof of the Theorem

We give here a proof of Theorem 1, repeated here.

**Theorem 1**

For any simple forest of stars, $\mathcal{F}$, there exists a positive constant, $\Lambda$, such that if $\lambda \geq \Lambda$, then the "purging" transformation (Def. 5),

$$f_\lambda : \mathcal{F} \rightarrow E^n,$$

has the following properties

1. $f_\lambda | \mathcal{F}$ is a homeomorphism from $\mathcal{F}$ onto its purged version $\hat{\mathcal{F}}$;

2. $f_\lambda | (\mathcal{F} - \mathcal{C})$ is an analytic diffeomorphism, where $\mathcal{C} \subset \partial \mathcal{F}$ is thin in $\partial \mathcal{F}$, corresponding to "sharp corners";

3. the destination point, $q_d$, is a fixed point of $f_\lambda$: $f_\lambda(q_d) = q_d$;

4. the Jacobian of $f_\lambda$ is bounded on $\mathcal{F} - \mathcal{C}$, provided that the gradient of the "obstruct functions", $\nabla \delta_i$ for $i \in I$, is bounded on $\mathcal{F} - \mathcal{C}$.

Proof: According to Proposition 3.1, in order to show that $f_\lambda$ is a homeomorphism, we first have to show that $f_\lambda$ is continuous on $\mathcal{F}$ and of class $C^q$ for $q \geq 1$ on $\mathcal{F} - \mathcal{C}$. This is true by construction, as we argue in detail in [14, Theorem 1]: namely, we show that $f_\lambda$ is continuous on $\mathcal{F}$ and analytic on $\mathcal{F} - \mathcal{C}$.

Next we have to show that $f_\lambda$ maps $\partial_j \mathcal{F}$ into $\partial_j \hat{\mathcal{F}}$ for $j \in \{0, \ldots, M\}$. According to the definition of $\hat{\mathcal{F}}$ (eq. 13), a point $p \in \partial_j \hat{\mathcal{F}}$ is either in the portion of $\partial \mathcal{F}$ which is also a part of $\partial \mathcal{F}$, namely, $\partial \mathcal{O}_i \cap \mathcal{F}$ for some $i \in I - \mathcal{L}$; or in a parent's boundary exposed by the "purging" of $\mathcal{F} - \overline{\mathcal{O}_i} \cap \partial \mathcal{O}_i \cap \mathcal{F}$ for some $i \in \mathcal{L}$; while any point $q \in \partial_j \mathcal{F}$ is in $\partial \mathcal{O}_i \cap \mathcal{F}$ for some $j \in I$. As we have discussed in Section 1.3 (after Def. 5), $f_\lambda$ becomes the identity mapping on $\partial \mathcal{O}_i \cap \mathcal{F}$ for $i \in I - \mathcal{L}$, and is the "ray scaling" map $f_i$ on $\partial \mathcal{O}_i \cap \mathcal{F}$ for $i \in \mathcal{L}$. Thus it suffices to consider $f_\lambda | (\partial \mathcal{O}_i \cap \mathcal{F})$ for $i \in \mathcal{L}$.

$$f_\lambda | (\partial \mathcal{O}_i \cap \mathcal{F}) = \hat{f}_i(q) = \hat{\nu}_i(q)(\tilde{q} - \tilde{q}_i) + \tilde{q}_i,$$

where $\hat{\nu}_i = \nu_i | \partial \mathcal{O}_i$. Let $q$ be a point in $\partial \mathcal{O}_i \cap \mathcal{F}$ for some $i \in \mathcal{L}$. Consider the ray from $q_i$ through $q$, $r_q(s) = s(q - q_i) + q_i$ for $s > 0$. According to [14, Lemma 1.5]

$$0 < \hat{\nu}_i \leq 1 \text{ for all } q \in \partial \mathcal{O}_i \cap \mathcal{F},$$

therefore $f_i(q)$ is located along the (image of) $r_q$ between $q_i$ and $q$, so that $f_i(q) = r_q(s_q)$ for some $0 < s_q \leq 1$. According to [14, Prop. 2.1], if $\mathcal{O}_i$ is strictly star shaped, then each ray from $q_i$, the center of $\mathcal{O}_i$, through a point $q \in \partial \mathcal{O}_i$ crosses $\partial \mathcal{O}_i$ exactly at $q$. Therefore, since $\mathcal{O}_i$ is strictly star shaped with respect to $q_i$, $r_q[0, 1] \subset \overline{\mathcal{O}_i}$, and as a consequence $f_i(q) \in \overline{\mathcal{O}_i}$. We show now that $f_i(q) \in \pi_1(0)$ as well, where

$$\pi_i(q) = \left\{ \begin{array}{ll}
(q - p_i) \cdot \nu_i - c_i & \text{if } P_i \text{ is planar} \\
\pm \left[ (q - p_i)^T Q_i (q - p_i) - 1 \right] & \text{if } P_i \text{ is quadratic}
\end{array} \right.$$
is the polynomial whose zero level set contains the patch \( \mathcal{P}_i \). To do this, let us rewrite \( \tilde{v}_i \),

\[
\tilde{v}_i(q) = \begin{cases} 
-\frac{\pi_i(q_i)}{\delta_i(q)} & \mathcal{P}_i \text{ planar} \\
\frac{(q - q_i)^T Q_i (p_i - q_i) + \text{sgn}(D^2 \pi_i) \sqrt{\delta_i(q)}}{(q - q_i)^T Q_i (q - q_i)} & \mathcal{P}_i \text{ quadratic}. 
\end{cases}
\]

(Eq. 26)

Evaluating \( \pi_1 \) at \( f_i(q) \) for a planar patch yields

\[
(\pi_1 \circ f_i)(q) = (f_i(q) - p_i) \cdot v_i - c_i = -\frac{\pi_i(q_i)}{\delta_i(q)} \frac{(q - q_i) \cdot v_i + (q_i - p_i) \cdot v_i - c_i}{\delta_i(q)} = 0,
\]

while for a quadratic patch,

\[
(\pi_1 \circ f_i)(q) = (f_i(q) - p_i)\cdot Q_i (f_i(q) - p_i) - 1 \\
= (\tilde{v}_i(q - q_i) + q_i - p_i)\cdot Q_i (\tilde{v}_i(q - q_i) + q_i - p_i) + 2\tilde{v}_i(q - q_i)\cdot Q_i (q_i - p_i) + (q_i - p_i)\cdot Q_i (q_i - p_i) - 1
\]

Since \((q - q_i)^T Q_i (q - q_i) > 0\) on \( \mathcal{F} \), we can factor the last equation as follows,

\[
(\pi_i \circ f_i)(q) = (\tilde{v}_i - \frac{(q - q_i)^T Q_i (p_i - q_i) + \sqrt{\delta_i(q)}}{(q - q_i)^T Q_i (q - q_i)}) (\tilde{v}_i - \frac{(q - q_i)^T Q_i (p_i - q_i) - \sqrt{\delta_i(q)}}{(q - q_i)^T Q_i (q - q_i)}) = 0,
\]

according to the formula for \( \tilde{v}_i \) (Eq. 26). Thus we have that

\[
f_i(q) \in \pi_i^{-1}(0) \cap \overline{\mathcal{C}_i} \text{ for all } q \in \partial \mathcal{C}_i \cap \mathcal{F}.
\]

We still have to show that \( f_i(q) \in \partial \mathcal{C}_i \cap \overline{\mathcal{C}_i} = \mathcal{P}_i \cap \overline{\mathcal{C}_i} \), for \( \pi_i^{-1}(0) \) might have points outside \( \partial \mathcal{C}_i \). By construction, \( q_i \in \mathcal{C}_i \) as well, therefore, since \( \partial \mathcal{C}_i \) separates \( \mathbb{R}^n \) and \( q \in \partial \mathcal{C}_i \cap \mathcal{F} \subset \mathbb{R}^n - \mathcal{C}_i \), \( r_q \) crosses \( \partial \mathcal{C}_i \) at some point \( q_i = r_q(s_i) > 0, s_i < 1 \). Since \( r_q[0,1] \subset \overline{\mathcal{C}_i} \), we have that \( q_i \in \partial \mathcal{C}_i \cap \overline{\mathcal{C}_i} \); and since, by construction,

\[
\partial \mathcal{C}_i \cap \overline{\mathcal{C}_i} \subset \mathcal{P}_i \cap \overline{\mathcal{C}_i},
\]

we have that \( q_i \in \mathcal{P}_i \cap \overline{\mathcal{C}_i} \). Now, if \( f_i(q) \notin \mathcal{P}_i \cap \overline{\mathcal{C}_i} \), then \( f_i(q) \notin \mathcal{P}_i \). We will show that for both types of patches this is impossible; the ray \( r_q \) cannot cross \( \mathcal{P}_i \) twice while inside \( \overline{\mathcal{C}_i} \). First, for a planar patch, this is impossible because \( r_q \) can cross the plane \( \pi_i^{-1}(0) \) only once, while we have that \( f_i(q) \in \pi_i^{-1}(0) \) (Eq. 27) and \( q_i \in \mathcal{P}_i \subset \pi_i^{-1}(0) \). For a quadratic patch, we first compute \( \nabla \pi_i \cdot (q - q_i) \) at \( f_i(q) \),

\[
(\nabla \pi_i \circ f_i(q)) \cdot (f_i(q) - q_i) = \tilde{v}_i(q) (\nabla \pi_i \circ f_i(q)) \cdot (q - q_i) = \tilde{v}_i, \text{sgn}(D^2 \pi_i) (f_i(q) - p_i)^T Q_i (q - q_i)
\]

\[
= \tilde{v}_i, \text{sgn}(D^2 \pi_i) (\tilde{v}_i(q - q_i) + (q_i - p_i)^T Q_i (q - q_i) + (q_i - p_i)(q - q_i)).
\]

Expanding \( \tilde{v}_i \) for a quadratic patch (Eq. 26) yields

\[
(\nabla \pi_i \circ f_i(q)) \cdot (f_i(q) - q_i)
\]

\[
= \tilde{v}_i(q) \text{sgn}(D^2 \pi_i) \left( -(q_i - p_i)^T Q_i (q - q_i) + \text{sgn}(D^2 \pi_i) \sqrt{\delta_i(q)} + (q_i - p_i)^T Q_i (q - q_i) \right)
\]

\[
= \tilde{v}_i(q) \text{sgn}(D^2 \pi_i) \sqrt{\delta_i(q)} > 0.
\]
since according to [14, Lemma C.5], $\tilde{v}_i > 0$ on $\partial \mathcal{O}_i \cap \mathcal{F}$. There is no problem with $\sqrt{\tilde{v}_i q_i}$, since by construction $\delta_i^{-1}(0, \infty)$ is a neighborhood about $\partial \mathcal{O}_i \cap \mathcal{F}$ in $\mathbb{R}^n$ [14, Lemma 2.2]. We also know that $\nabla \pi_i(q_i) \cdot (q_i - q_i) > 0$, since, by construction,

$$\nabla \pi_i(q) \cdot (q - q_i) > 0 \quad \text{for all } q \in \mathcal{P}_i,$$

and $q_i \in \mathcal{P}_i$. Thus we have that both $q_1$ and $f_i(q)$ are in $\pi_i^{-1}(0)$, such that

$$\frac{d}{ds} \big|_{s = s_0} (\pi_i \circ r_q)(s) > 0 \quad \text{and} \quad \frac{d}{ds} \big|_{s = s_1} (\pi_i \circ r_q)(s) > 0.$$

Since $\pi_i$ experiences the same sign change along $r_q$ at $q_1$ and $f_i(q)$, it must be that $r_q$ crosses $\pi_i^{-1}(0)$ at least in a three distinct points — an impossibility, since $\pi_i$ is quadratic. We conclude that $f_i(q) \in \mathcal{P}_i \cap \overline{\mathcal{O}_i}$, and as a consequence that

$$f_i(\partial \mathcal{O}_i \cap \mathcal{F}) \subset \mathcal{P}_i \cap \overline{\mathcal{O}_i} \quad \text{for all } i \in \mathcal{L},$$

which implies that $f_i(\partial \mathcal{O}_i \cap \mathcal{F}) \subset \partial \mathcal{O}_i$ for $j \in \{0, \ldots, M\}$.

We now show that $f_i(\partial \mathcal{O}_i \cap \mathcal{F})$ is one-to-one for all $i \in \mathcal{L}$. Since $f_i | \partial \mathcal{O}_i \cap \mathcal{F}$ is the identity mapping away from the leaves' boundary, this would imply that $f_i(\partial \mathcal{O}_i \cap \mathcal{F})$ is one-to-one for $j \in \{0, \ldots, M\}$. Suppose to the contrary, that there exist two points $q, q' \in \partial \mathcal{O}_i \cap \mathcal{F}$, such that $f_i(q) = f_i(q')$, that is,

$$\tilde{v}_i(q)(q - q_i) + q_i = \tilde{v}_i(q')(q' - q_i) + q_i,$$

or,

$$q' = \frac{\tilde{v}_i(q')}{\tilde{v}_i(q)} \cdot (q - q_i) + q_i.$$

According to [14, Lemma C.5], $\tilde{v}_i > 0$ on $\partial \mathcal{O}_i \cap \mathcal{F}$, therefore $q'$ is located along the ray $r_{q_i}$, an impossibility, since by hypothesis, $\mathcal{O}_i$ is strictly star shaped and according to [14, Prop. 2.1] any such ray crosses $\partial \mathcal{O}_i$ exactly once.

We now show that $f_i$ maps $\partial \mathcal{O}_i \cap \mathcal{F}$ onto $\mathcal{P}_i \cap \overline{\mathcal{O}_i}$, for $i \in \mathcal{L}$. Consider the ray $r_p(s) = s(p - q_i) + q_i$, with $s > 0$, for some point $p \in \mathcal{P}_i \cap \overline{\mathcal{O}_i}$. Using [14, Prop. 2.2] again, $r_p$ crosses $\partial \mathcal{O}_i$ at a unique point $q = r_p(s_0)$ for some $s_0 > 0$. We have to show that $q \in \partial \mathcal{O}_i \cap \mathcal{F}$. First, it is impossible that $0 < s_0 < 1$, for this would imply that $r_p$ crosses $\partial \mathcal{O}_i$ before reaching $p$, and therefore $p \notin \mathcal{F}$, contradicting our hypothesis that $p \in \mathcal{P}_i \cap \overline{\mathcal{O}_i}$. Thus it must be that $s_0 \geq 1$. Suppose to the contrary, that $q \notin \partial \mathcal{O}_i \cap \mathcal{F}$. By construction,

$$\partial \mathcal{O}_i = \left( \partial \mathcal{O}_i \cap \mathcal{F} \right) \cup \left( \partial \mathcal{O}_i \cap \mathcal{O}_p(0) \right),$$

a disjoint union, therefore $q \notin \partial \mathcal{O}_i \cap \mathcal{O}_p(0)$. Since $p \in \mathcal{P}_i \cap \overline{\mathcal{O}_i} \subset \partial \mathcal{O}_p(0)$, and $\mathcal{O}_p(0) \subset \mathcal{O}_p(0) \subset \mathcal{O}_p(0) \subset \mathcal{O}_p(0)$, we have that $p \notin \mathcal{F}$. Furthermore, $\mathcal{O}_p(0)$ separates $\mathbb{R}^n$ and both $q$ and $q$ are in $\mathcal{O}_p(0)$, therefore $r_p$ crosses $\partial \mathcal{O}_p(0)$ at $p$ and then at least once more, at some point $p_1 = r_p(s_1)$, before reaching $q$. Since $r_p$ starts at $q_i \subset \mathcal{O}_i$, and crosses $\partial \mathcal{O}_i$ only at $q$, both $p$ and $p_1$ are inside $\overline{\mathcal{O}_i}$. By construction,

$$\partial \mathcal{O}_p(0) \cap \overline{\mathcal{O}_i} \subset \mathcal{P}_i = \pi_i^{-1}(0) \cap \partial \mathcal{O}_p(0).$$
therefore both \( p \) and \( p_1 \) are in \( \pi_i^{-1}(0) \), and as a consequence,

\[
\frac{d}{ds} \mid_{s=s_1} (\pi_i \circ r_p)(s) > 0 \quad \text{and} \quad \frac{d}{ds} \mid_{s=1} (\pi_i \circ r_p)(s) > 0,
\]

which is impossible for a planar or quadratic polynomial. Thus we have proved that for all \( \mu \in \mathcal{P}_i \cap \overline{\gamma}_i \), the ray \( r_\mu \) crosses \( \partial \Omega_i \) at a point \( q \in \partial \Omega_i \cap \mathcal{F} \), and therefore \( f_i \) maps \( \partial \Omega_i \cap \mathcal{F} \) onto \( \mathcal{P}_i \cap \overline{\gamma}_i \). We conclude that \( f_\lambda \) maps \( \partial_j \mathcal{F} \) onto \( \partial_j \hat{\mathcal{F}} \) for \( j \in \{0, \ldots, M\} \).

Summing up, \( f_\lambda \mid \partial_j \mathcal{F} \) is a bijection onto \( \partial_j \hat{\mathcal{F}} \) for \( j \in \{0, \ldots, M\} \), and has a non-singular Jacobian on \( \mathcal{F} - \mathcal{C} \). According to Proposition 3.1, \( f_\lambda \) is a homeomorphism from \( \mathcal{F} \) onto \( \hat{\mathcal{F}} \). Since we have shown that \( f_\lambda \) is analytic and has a non-singular Jacobian on \( \mathcal{F} - \mathcal{C} \), it follows from the Inverse Function Theorem and the fact that \( f_\lambda \) is one-to-one on \( \mathcal{F} \), that \( f_\lambda \) is an analytic diffeomorphism on \( \mathcal{F} - \mathcal{C} \).

To check that \( f_\lambda(q_\alpha) = q_\alpha \), recall that the term \( \gamma_\alpha = \|q - q_\alpha\|^2 \) appears as a factor in all the switches, \( \sigma_i \) for \( i \in \mathcal{L} \). Thus,

\[
\sigma_i(q_\alpha) = 0 \quad \text{for all} \quad i \in \mathcal{L},
\]

and as a consequence \( \sigma_i(q_\alpha) = 1 - \sum_{i \in \mathcal{L}} \sigma_i(q_\alpha) = 1 \).

Finally, we show in [14, Lemma C.12] that \( D f_\lambda \) is bounded on \( \mathcal{F} - \mathcal{C} \), provided that the gradient of the obstacle functions, \( \nabla \beta_i \) for \( i \in \mathcal{I} \), are bounded on \( \mathcal{F} - \mathcal{C} \).

\[
\square
\]

B.3 The Jacobian of \( f_\lambda \) is Non-Singular

Consider the set \( \mathcal{S}_i(\epsilon) \) the "\( \epsilon \)-thickened" \( i^{th} \) star's boundary,

\[
\mathcal{S}_i(\epsilon) = \{ q \in \mathcal{F} : 0 \leq \theta_i \leq \epsilon \},
\]

with the constraint \( \epsilon \leq \epsilon_i \). It follows from the definition of \( \epsilon_i \) (specifically, the "half gaps" in Def. 8) that \( \epsilon \) is small enough to guarantee that the sets \( \mathcal{S}_i(\epsilon) \) for \( i \in \mathcal{L} \) are disjoint and do not overlap the destination point.

We distinguish in the forest \( \mathcal{F} \) the set "away from the leaves",

\[
\mathcal{A}(\epsilon) \triangleq \bigcap_{i \in \mathcal{L}} \{ q \in \mathcal{F} : \beta_i(q) > \epsilon \},
\]

and denote its complement in \( \mathcal{F} \) by

\[
\mathcal{A}'(\epsilon) \triangleq \mathcal{F} - \mathcal{A}(\epsilon) = \bigcup_{i \in \mathcal{L}} \mathcal{S}_i(\epsilon),
\]

a disjoint union.
We are ready to prove the non-singularity of the Jacobian on the set "c away" from the leaves, \( \Lambda(c) \). In [14] we prove that for any \( \epsilon > 0 \) and any \( \delta > 0 \), \( \sigma_i(q, \lambda) \) and \( \| \lambda \| \leq \delta \) can be made to be smaller than \( \delta \) on the set "c away" from the \( i^{th} \) obstacle, \( F - S_i(c) \), by choosing
\[
\lambda(c, \delta) \geq N_0(c, \delta) \quad \text{and} \quad \lambda(c, \delta) \geq N_1(c, \delta),
\]
respectively, where \( N_0(c, \delta), N_1(c, \delta) \) are positive constants defined in terms of the geometrical data. As a consequence, it is possible to guarantee that for any fixed \( \epsilon > 0 \), the Jacobian of \( f_\lambda \) is non-singular on \( \Lambda(c) \), a statement which is made precise as follows.

**Lemma B.2** Given a simple forest of stars \( F \), for any \( \epsilon > 0 \) there exists a positive constant \( \Lambda_0(\epsilon) \) such that if \( \lambda \geq \Lambda_0 \) then \( Df_\lambda \) is non-singular on \( \Lambda(c) - C \), where \( C \subset \partial F \) is thin (nowhere dense) in \( \partial F \).

The proof is given in [14].

Consider now the complementary portion of \( F - C \), the set \( \Lambda'(c) - C \). \( Df_\lambda(q) \) maps \( i_q, j_q \) into \( T_{\lambda(q)} f_\lambda(F) \), both of which are \( n \)-dimensional vector spaces, and therefore isomorphic to \( E^n \). Since the domain and image of \( f_\lambda \) the forest, \( F \), and its purged version, \( \tilde{F} \) are both \( n \)-dimensional submanifolds of \( E^n \), we will not distinguish points in the tangent spaces from points in the original base spaces.

Let \( (q - q_i) \) be a vector based at \( q \in E^n \). Consider the tangent space to \( F \) at \( q, T_q F \), as the orthogonal direct sum
\[
T_q F = \langle q - q_i \rangle \oplus \langle q - q_i \rangle^\perp,
\]
where \( \langle \cdot \rangle \) denotes the "span of", and \( (\cdot)^\perp \) denotes the "orthogonal complement of". Each vector \( x \in T_q F \) can be uniquely written as
\[
x = x_1 + x_2 \quad \text{such that} \quad x_1 \in \langle q - q_i \rangle \quad \text{and} \quad x_2 \in \langle q - q_i \rangle^\perp.
\]
(29)

We are ready to prove the existence of a neighborhood about the leaves' boundary in \( \tilde{F} - C \), in which \( Df_\lambda \) is non-singular.

**Proposition B.3** Given a simple forest of stars \( F \), for each \( i \in T \) there exist positive constants, \( \lambda_i \) and \( \epsilon_i \), such that if \( \lambda \geq \lambda_i \) then the Jacobian of \( f_\lambda \) is non-singular in the set
\[
S_i(\epsilon_i) - C \equiv \{ q \in F : 0 \leq \beta_i(q) \leq \epsilon_i \} - C,
\]
where \( C \subset \partial F \) is thin in \( \partial F \).

The proof of is given in [14]. Since we have considered the entirety of \( F - C \), the above results are summarized in the following Corollary.

**Corollary B.4** Given a simple forest of stars \( F \), there exists a positive constant, \( \Lambda \), such that if \( \lambda \geq \Lambda \), then the Jacobian of \( f_\lambda \) is non-singular in \( F - C \).
B.3 The Jacobian of $f_\lambda$ is Non-Singular

Proof: Using Proposition B.3, we choose a constant,

$$
\epsilon^* \triangleq \min_{i \in \mathcal{L}} \{ \epsilon_i \},
$$

designating a neighborhood about the leaves' boundary in $\mathcal{F} - \mathcal{C}$. $\mathcal{A}(\epsilon^*) - \mathcal{C}$, in which $Df_\lambda$ is non-singular, whenever the parameter $\lambda$ satisfies

$$
\lambda \geq \max_{i \in \mathcal{L}} \{ \Lambda_i \} \triangleq \Lambda_1.
$$

According to Lemma B.2, for any $\epsilon^* > 0$, if the parameter $\lambda$ satisfies

$$
\lambda \geq \Lambda_0(\epsilon^*),
$$

then $Df_\lambda$ is non-singular in the set $\mathcal{A}(\epsilon^*)$.

Thus, letting

$$
\Lambda \triangleq \max \{ \Lambda_0(\epsilon^*), \Lambda_1 \},
$$

completes the proof.

\[ \square \]

Evidently, we have postponed the details of the proof to [14]. It is important to point out that during the proof of the non-singularity of the Jacobian of $f_\lambda$, an explicit formula for the computation of the parameter $\lambda$ is derived in terms of the geometrical data, thus making Theorem 1 practicable. In the next section we discuss the computational cost of the parameter $\lambda$ and the geometrical data required.
References


