Technical Report on:
Comparative Design, Scaling, and Control of Appendages for Inertial Reorientation

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This technical report provides full derivations and definitions for the paper, [1].

I. ANALYTIC SOLUTION OF TEMPLATE DYNAMICS

Integration of the system dynamics, given in [1, Eqn. 8] as,

\[
\dot{\theta}_b = \begin{cases} 
\frac{4P}{\omega_m I_d} \left( 1 - \frac{\theta_b}{\xi \omega_m} \right), & \text{for } 0 \leq t < t_s, \\
-\frac{4P}{\omega_m I_d}, & \text{for } t \geq t_s,
\end{cases}
\]

is easier in the rescaled coordinates introduced in [1, Eqn. 13],

\[
\tilde{t}_s = \gamma t_s, \quad \tilde{t}_f = \gamma t_f, \quad \tilde{\theta}_h = \frac{\theta_h}{\theta_{h,f}}, \quad \tilde{\omega}_m = \frac{\xi \omega_m}{\gamma \theta_{h,f}}
\]

where [1, Eqn. 14],

\[
\gamma := \left( \frac{4P \xi}{I_d \theta_{h,f}} \right)^{\frac{1}{4}}.
\]

We will use prime notation instead of a dot to denote time derivatives with respect to \( \tilde{t} \), i.e. \( \dot{} := d/d\tilde{t} \),

\[
\dot{\theta}_b = \gamma \theta_{h,f} \dot{\theta}_b', \quad \tilde{\theta}_h = \gamma^2 \theta_{h,f} \tilde{\theta}_h' .
\]

In the rescaled system, the dynamics are simply,

\[
\ddot{\theta}_b' = \begin{cases} 
\frac{1}{\omega_m} \left( 1 - \frac{\dot{\theta}_b'}{\omega_m} \right), & \text{for } 0 \leq \tilde{t} < \tilde{t}_s, \\
-\frac{1}{\omega_m}, & \text{for } \tilde{t} \geq \tilde{t}_s.
\end{cases}
\]

Integrating from the initial conditions \( \dot{\theta}'(0) = 0 \) and \( \tilde{\theta}(0) = 0 \) yields the flow over the acceleration phase,

\[
\dot{\theta}'(\tilde{\omega}_m, \tilde{t}) = \tilde{\omega}_m \left( 1 - \exp \left( -\frac{\tilde{t}}{\tilde{\omega}_m^2} \right) \right)
\]

(5)

\[
\tilde{\theta}(\tilde{\omega}_m, \tilde{t}) = \tilde{\omega}_m t - \tilde{\omega}_m^2 \left( 1 - \exp \left( -\frac{\tilde{t}}{\tilde{\omega}_m} \right) \right),
\]

(6)

for \( \tilde{t} < \tilde{t}_s \). The flow over deceleration is,

\[
\dot{\theta}'(\tilde{\omega}_m, \tilde{t}) = \dot{\theta}'(\tilde{\omega}_m, \tilde{t}_s) - \frac{1}{\tilde{\omega}_m} (\tilde{t} - \tilde{t}_s),
\]

(7)

\[
\tilde{\theta}(\tilde{\omega}_m, \tilde{t}) = (\tilde{t} - \tilde{t}_s) \dot{\theta}'(\tilde{\omega}_m, \tilde{t}_s) - \frac{1}{2\tilde{\omega}_m} (\tilde{t} - \tilde{t}_s)^2 + \dot{\tilde{\theta}}(\tilde{\omega}_m, \tilde{t}_s),
\]

(8)

for \( \tilde{t} > \tilde{t}_s \). The maneuver ends at a halting time \( \tilde{t}_h = \tilde{t}_s + \tilde{t}_r \), when the body comes to rest. The duration of the braking phase, \( \tilde{t}_r \), is the zero of (7), or equivalently the speed at the switch divided by the braking acceleration \( 1/\tilde{\omega}_m \),

\[
\tilde{t}_r = \tilde{\omega}_m \dot{\tilde{\theta}}'(\tilde{\omega}_m, \tilde{t}_s).
\]

(9)

The final body angle is thus an explicit function of the switching time and \( \tilde{\omega}_m \), and can be written out by combining (5)–(9),

\[
\tilde{\theta}_h = \tilde{\theta}_h(\tilde{\omega}_m, \tilde{t}_s) := \tilde{\theta}(\tilde{t}_s + \tilde{t}_r, \tilde{\omega}_m, \tilde{\omega}_m, \tilde{t}_s) = \tilde{t}_r \dot{\tilde{\omega}}(\tilde{\omega}_m, t_s) - \frac{1}{2\tilde{\omega}_m^2} (\tilde{t}_s - \tilde{t})^2 + \dot{\tilde{\theta}}(\tilde{\omega}_m, t_s)
\]

(10)

The halting time is simply the sum of the switching time and the braking time,

\[
\tilde{t}_h = \tilde{\theta}_h(\tilde{\omega}_m, \tilde{t}_s) := \tilde{t}_s + \tilde{t}_r = \tilde{t}_s + \tilde{\omega}_m^2 \left( 1 - \exp \left( -\frac{\tilde{t}_s}{\tilde{\omega}_m} \right) \right)
\]

(11)
Substituting the definitions of the rescaled coordinates, (1) and (2), into (10) and (11),
\[ t_h = g_h(p) := \frac{\tilde{\gamma}_b}{\gamma_{\beta_b, \gamma}} \left( \frac{\xi \omega_m}{\gamma_{\beta_b, \gamma}} t_s \right) \]
(12)
\[ = t_s + \frac{I_b \xi \omega_m^2}{4P} \left( 1 - \exp \left( - \frac{4P}{I_b \xi \omega_m^2} t_s \right) \right), \]
(13)
\[ \theta_h = g_\theta(p) := \theta_{b, \gamma} \tilde{\gamma}_b \left( \frac{\xi \omega_m}{\gamma_{\beta_b, \gamma}} t_s \right) \]
(14)
\[ = \xi \omega_m t_s - \frac{I_b \xi \omega_m^3}{8P} \left( 1 - \exp \left( - \frac{8P}{I_b \xi \omega_m^2} t_s \right) \right). \]
(15)
as given in [1, Eqn. 9] and [1, Eqn. 10].

A. Analytic solution of template dynamics with a current limit

If the maximum allowable torque is limited to some factor \( \beta \in (0, 1) \) less than the stall torque of the motor, \( \tau_f = \tau_{m_f} \), the optimal reorientation consists of three phases: a constant torque phase until a time \( \bar{t}_i \) when the acceleration becomes voltage-limited, then a phase following the speed–torque curve of the motor until the controlled switch at \( t_s \), followed by a constant braking torque phase of duration \( \bar{t}_s \) until \( t_h \). In this case, the time-switched dynamics of (4) become instead,
\[ \tilde{\beta}(\omega_m, \tilde{t}, \beta) = \tilde{\beta}(\omega_m, \tilde{t_s}, \beta) - \beta \omega_m \]
(24)
\[ \tilde{\theta}(\omega_m, \tilde{t}, \beta) = \tilde{\theta}(\omega_m, \tilde{t_s}, \beta) + (\tilde{t} - \tilde{t}_s) \tilde{\theta}(\omega_m, \tilde{t_s}, \beta) \]
(25)
for \( \tilde{t}_s \leq \tilde{t} \leq \tilde{t}_s \). Finally, the flow over deceleration is,
\[ \tilde{\theta}(\omega_m, \tilde{t}, \beta) = \tilde{\theta}(\omega_m, \tilde{t_s}, \beta) \]
(26)
for \( \tilde{t} > \tilde{t}_s \). The analysis follows similarly to the previous section, with the return time given by the function,
\[ \tilde{t}_r = \tilde{g}_r(\omega_m, \tilde{t_s}, \beta) := \frac{\omega_m}{\beta} \tilde{\theta}(\omega_m, \tilde{t_s}, \beta), \]
(27)
the final time given by,
\[ \tilde{t}_h = \tilde{g}_h(\omega_m, \tilde{t_s}, \beta) := \tilde{t}_s + \tilde{t}_r \]
(28)
and the explicit form of \( \tilde{g}_\theta \),
\[ \tilde{\theta}_h = \tilde{g}_\theta(\omega_m, \tilde{t_s}, \beta) := \tilde{\theta}(\omega_m, \tilde{t_s} + \tilde{g}_r(\omega_m, \tilde{t_s}, \beta)), \]
(29)
\[ = \tilde{\theta}(\omega_m, \tilde{t_s}) + \tilde{\omega}_m \beta \left[ \left( \tilde{\theta}(\omega_m, \tilde{t_s}) \right)^2 \right] \]
(30)
\[ = \omega_m \tilde{t}_s + \tilde{\omega}_m \beta - 1 \exp \left( \frac{1 - \beta}{\beta} - \frac{\tilde{t}_s - \tilde{t}_l}{\tilde{\omega}_m} \right) \]
(31)
Note that if \( \tilde{\omega}_m \) is very large, the acceleration will be so slow that the system never reaches the speed-limited phase and the critical switching time \( t_c \geq t_l \). In this case, the acceleration and braking phases are symmetric with equal durations and \( t_h = 2t_s \). The condition for this behavior can be found by taking \( \tilde{t}_s = \tilde{t}_l \) in (30) and is,
\[ \tilde{\omega}_m \geq \left( \frac{\beta}{(1 - \beta)^2} \right)^{\frac{1}{3}}. \]
(32)
The optimal gearing can be found by using the critical switching time, \( t_c = \tilde{g}_c(\omega_m) \) and minimizing the final time \( \tilde{t}_h = \tilde{g}_h(\omega_m, \tilde{t}_s, \beta) \), with no constraints on \( \tilde{\omega}_m \) other than non-negative real. The halting time \( \tilde{t}_h \) varies with \( \beta \), thus varying the power constant \( k_p \) and speed constant \( k_s \), as defined in [1, Sec. II-C1]. Optimal gear ratio is only weakly sensitive to current limit, varying less than 5% over the possible values of \( \beta \) (Fig. 1, middle). The required nominal power with the optimal gear ratio grows rapidly with decreasing current limit; limiting torque to 50% increases required nominal power by 53%, while a current limit of 25% nearly triples the required nominal power (Fig. 1, bottom).

The current-limited versions of the template behavior, [1, Eqn. 9] and [1, Eqn. 10], can be derived by substituting the definitions of the rescaled coordinates, (1), into (28) and (30) as in the previous subsection.

II. ALTERNATE CONTROLLER FORMULATIONS

A. Event-based switching

The time-switched bang-bang controller of the previous section can be replaced by an event-based switch or guard
condition $G(\theta, \theta') = 0$. For example,

$$G_\theta := \ddot{\theta} - \ddot{\theta}_s,$$

where the value of $\ddot{\theta}_s$ is found easily from (6) (or (23) for the current-limited case). For the optimally geared case with $\beta = 1$, $\theta_s \approx 0.7$; this value changes for suboptimal gearing or $\beta < 1$ (Fig. 1, top).

### B. Feedback controllers regulating body angle

For additional robustness, the template controller may use proportional-derivative (PD) feedback on the body angle (relative to the desired final position, $\theta_{b,f}$, and velocity, $\theta_b = 0$). The controller torque takes the form,

$$\tau = K_p(\theta_{b,f} - \theta_b) + K_d(0 - \dot{\theta}_b),$$

subject to the limits imposed by the motor model. Given high enough gains, the torque will saturate, producing speed-limited acceleration and current-limited braking as in the switched case; the effective switching time (when $\tau = 0$) depends on the ratio of controller gains. Finding the ratio of gains corresponding to a particular value of $p$ is easily done in the dimensionless system coordinates. Substituting $\tau = \ddot{\theta}_b/I_d$ and applying the spatiotemporal rescaling of (1) to (34), the dimensionless closed-loop dynamics are,

$$\theta'' = \tilde{K}_p(1 - \tilde{\theta}) + \tilde{K}_d(0 - \tilde{\theta'}),$$

where $\tilde{K}_p = K_p/(\gamma^2 I_d)$ and $\tilde{K}_d = K_d/(\gamma I_d)$. The closed loop dynamics are subject to the motor-imposed acceleration limits,

$$-\frac{\beta}{\omega_m} \leq \ddot{\theta} \leq \frac{\beta}{\omega_m}(1 - \frac{\ddot{\theta}'}{\omega_m}),$$

for $\ddot{\theta}' \geq 0$ (the condition for negative body velocity is found by multiplying the inequality by $-1$, but will never occur during the optimal reorientation).

Substituting expressions (5)–(6) for the state at the time of switch (where $\theta'' = 0$),

$$0 = \tilde{K}_p(1 - \tilde{\theta}(\tilde{t}_s, \tilde{\omega}_m)) - \tilde{K}_d\tilde{\theta}'(\tilde{t}_s, \tilde{\omega}_m)$$

$$\tilde{K}_d = \frac{1 - \tilde{\theta}(\tilde{t}_s, \tilde{\omega}_m)}{\tilde{\theta}'(\tilde{t}_s, \tilde{\omega}_m)}$$

$$\tilde{K}_p = \frac{1 - \tilde{\omega}_m \tilde{\theta}_e + \tilde{\omega}_m^3(1 - \exp(-\frac{\tilde{t}_s}{\omega_m}))}{\tilde{\omega}_m(1 - \exp(-\frac{\tilde{t}_s}{\omega_m}))},$$

for $\beta = 1$ (the expression for the gain ratio follows the above, substituting (24)–(25) instead). The critical value that produces the optimal switch is found by substituting the optimal no-load speed, $\tilde{\omega}_m^* \approx 0.74$, and corresponding switching time, $\tilde{t}_s^* \approx 1.63$ and has a value of $\tilde{K}_d/\tilde{K}_p \approx 0.26$. For current-limited dynamics, the ratio of gains increases with decreasing $\beta$.

When scaling back to physical torques, the gains will scale with $I_d$ and $\gamma$ as defined above, so the optimal ratio is,

$$\frac{K_d}{K_p} = \frac{\tilde{K}_d}{\tilde{K}_p} \approx 0.26 \frac{1}{\gamma}$$

### III. DERIVATION OF TAIL CONNECTION FIELD

The angular momentum of the system of rigid bodies can be found by adding the angular momentum of each body with respect to some point, $O$,

$$\mathbf{H}_O = \mathbf{H}_{b,O} + \mathbf{H}_{t,O}.$$ 

Let $\{\mathbf{E}_1, \mathbf{E}_2\}$ be the world reference frame in the plane, and define $\mathbf{E}_3 := \mathbf{E}_1 \times \mathbf{E}_2$ which exits the page. Let $\mathbf{r}_b$ and $\mathbf{r}_t$ be the position vectors relative to $O$ of the body and tail, respectively (as in Fig. 2). Each link’s angular momentum is the sum of its angular momentum about its own COM and its moment of linear momentum about $O$,

$$\mathbf{H}_{i,O} = \int \mathbf{r}_i \mathbf{E}_3 + \mathbf{r}_i \times m_i \mathbf{v}_i,$$

where the subscripts $i \in \{b, t\}$ denote the body and tail, respectively, and $\mathbf{v}_i := \mathbf{v}_O + \mathbf{r}_i$ is the absolute velocity of each link. The total angular momentum of the two body system is,

$$\mathbf{H}_O = \mathbf{H}_{b,O} + \mathbf{H}_{t,O}$$

$$= (m_b \mathbf{r}_b + m_t \mathbf{r}_t) \times \mathbf{v}_O + \sum_{i \in \{b, t\}} \left( \int \mathbf{r}_i \mathbf{E}_3 + \mathbf{r}_i \times m_i \mathbf{v}_i \right).$$

(39)

The centroid of the combined tailed-body mechanism with respect to $O$, denoted $\mathbf{r}_{com}$, is a weighted sum of the link positions,

$$\mathbf{r}_{com} = \frac{m_b \mathbf{r}_b + m_t \mathbf{r}_t}{m_b + m_t}.$$ 

(40)
Hence the body and tail vectors are related by, \[ O \text{ angular momentum about } \text{COM}, \] and let \( \theta_b \) be the angle of \( e_r \) with respect to the world reference frame, i.e. the frame is defined by a rotation of \( \theta_a \) about \( E_3 \),

\[
e_r := \cos \theta_a E_1 + \sin \theta_a E_2; \tag{41}
\]

\[
e_s := -\sin \theta_a E_1 + \cos \theta_a E_2. \tag{42}
\]

This frame enables a simple definition of the vectors from the system COM to the segment COMs,

\[
r_b = re_r; \quad r_t = -(l-r)e_r. \tag{43}
\]

The definition of the center of mass fixes \( r \),

\[-mt(l-r) + mb r = 0 \quad \Rightarrow \quad r = \frac{m_t}{m_b + m_t}l. \tag{44}
\]

Hence the body and tail vectors are related by,

\[
r_t = \frac{r - l}{r}r_b = (1 - \frac{mb + mt}{mt})r_b = -\frac{mb}{mt}r_b. \tag{45}
\]

We can now simplify (39), the expression for total angular momentum,

\[
H_O = (I_b \dot{\theta}_b + I_d \dot{\theta}_d)E_3 + r_b \times (m_b \dot{r}_b) + r_t \times (m_t \dot{r}_t)
\]

\[
= (I_b \dot{\theta}_b + I_d \dot{\theta}_d)E_3 + \left( m_b + \frac{m_t^2}{m_t} \right) r_b \times \dot{r}_b. \tag{46}
\]

The last term of (46) describes the component of angular momentum due to the two point masses orbiting the COM. This cross product, derived below, is always perpendicular to the plane and has a relatively simple expression for its magnitude in terms of the body-fixed reference frame, \( \{e_r, e_s\} \), and the tail-fixed reference frame, \( \{e_{rt}, e_{st}\} \) (defined analogously to (41)–(42) as a rotation about \( E_3 \) of \( \theta_t \) and \( \theta_r \), respectively).

Equating two expressions for the vector from the pivot to the system COM,

\[
l_b e_{rb} - r_b = l_t e_{rt} - r_t
\]

\[
-\frac{m_b + m_t}{m_t} r_b = l_t e_{rt} - l_b e_{rb}
\]

\[
r_b = -\frac{m_t}{m_b + m_t} (l_t e_{rt} - l_b e_{rb}). \tag{47}
\]

The vector \( \dot{r}_b \) follows from time differentiation of \( r_b \),

\[
\dot{r}_b = -\frac{m_t}{m_b + m_t} (l_t \dot{e}_{rt} - l_b \dot{e}_{rb}) \tag{48}
\]

\[
= -\frac{m_t}{m_b + m_t} (l_t \dot{\theta}_t e_{st} - l_b \dot{\theta}_b e_{sb}). \tag{49}
\]

Hence the final term in (46) becomes,

\[
\frac{m_b m_t}{(m_b + m_t)} (l_t e_{rt} - l_b e_{rb}) \times (l_t \dot{\theta}_t e_{st} - l_b \dot{\theta}_b e_{sb}).
\]

The mass coefficient is also known as the reduced mass,

\[
m_r := \frac{m_b m_t}{(m_b + m_t)}. \tag{50}
\]

Using the following identities,

\[
(e_{rt} \times e_{sb}) = \cos \theta_r E_3; \quad (e_r \times e_{st}) = \cos \theta_r E_3, \tag{51}
\]

we can now evaluate the remaining cross product,

\[
(l_t e_{rt} - l_b e_{rb}) \times (l_t \dot{\theta}_t e_{st} - l_b \dot{\theta}_b e_{sb})
\]

\[
= (l_t^2 \dot{\theta}_t - l_b l_t \dot{\theta}_b \cos \theta_t - l_t l_b \dot{\theta}_t \cos \theta_r + l_b^2 \dot{\theta}_b) E_3
\]

\[
= ((l_t^2 - l_b l_t \cos \theta_r) \dot{\theta}_t + (l_b^2 - l_b l_t \cos \theta_r) \dot{\theta}_b) E_3.
\]

As all terms of \( H_O \) are perpendicular to the plane, we drop the vector rotation and simply examine the magnitude of the total angular momentum in this tail anchor, \( H_{O,t} \), where \( H_{O,t} E_3 = H_3 \). With the coordinate substitution, \( \theta_t = \theta_b + \theta_r \), and the simplification of the cross product,

\[
H_{O,t} = (I_b + m_r(l_b^2 - l_t l_b \cos \theta_r)) \dot{\theta}_b + (I_t + m_r(l_t^2 - l_b l_t \cos \theta_r)) \dot{\theta}_t
\]

\[
= (I_b + I_t + m_r(l_t^2 - 2l_t l_b \cos \theta_r)) \dot{\theta}_b + (I_t + I_r + m_r(l_b^2 - 2l_b l_t \cos \theta_r)) \dot{\theta}_t,
\]

as stated for the tail template kinematics, [1, Eqn. 30].

### A. Restriction on domain of dimensionless parameters

Because of coupling between the dimensionless constants and the requirement of non-negativity of the dimensioned parameters, only a subset of the dimensionless parameter space is physically realizable. By definition, [1, Eqn. 32], \( \xi_t \) is restricted to the interval \([0, 1]\), as the denominator is no smaller than the numerator and both are strictly positive. Furthermore, for a given value of \( \xi_t \) there is a maximum value of \( \eta \). Starting with positivity of physical parameters,

\[
0 < I_b I_t + I_b m_r l_t^2 + I_t m_r l_b^2
\]

\[
\frac{m_r^2 l_t^4 l_b^2}{(I_t + m_r l_t^2)^2} < \frac{I_b I_t + I_b m_r l_t^2 + I_t m_r l_b^2 + m_r^2 l_t^2 l_b^2}{(I_t + m_r l_t^2)^2}
\]

\[
\eta^2 < \frac{1 - \xi_t}{\xi_t}, \quad \eta < \sqrt{\frac{1 - \xi_t}{\xi_t}}, \tag{52}
\]

that is \( \eta \) is bounded above as shown in gray in [1, Fig. 5]. Another bound used in the paper, ensuring positivity of the denominator of [1, Eqn. 35], may be found by starting with the positivity of physical parameters and of squared values,

\[
0 < \frac{I_b + I_t + m_r (l_b - l_t)^2}{I_b + I_t + m_r (l_b^2 + l_t^2)}
\]

\[
0 < 1 - \frac{2m_r l_b l_t}{I_b + I_t + m_r (l_b^2 + l_t^2)} \quad 2\xi_t \eta < 1 \tag{53}
\]
B. Integration of the connection field

The total inertial effect of the tail over a given tail sweep, \( \theta_r \in [\theta_{r1}, \theta_{r2}] \), is the integral of the connection magnitude \( A(\theta_r) \), [1, Eqn. 35], over that stroke range. This integral can be written in closed form by first factoring the connection,

\[
A(\theta_r) = \frac{1}{2} \left( \frac{1 - 2 \xi \cos \theta_r}{1 - 2 \xi \eta \cos \theta_r} \right) = \frac{1}{2} \left( \frac{2 \xi - 1 - 2 \xi \eta \cos \theta_r}{2 \xi - 2 \xi \eta \cos \theta_r + 1 - 1} \right)
\]

The change in body angle, \( \Delta \theta_b := \theta_{b,f} - \theta_{b,0} \), over a given tail sweep, \( \Delta \theta_r := \theta_{r2} - \theta_{r1} \), is then,

\[
\Delta \theta_b = - \int_{\theta_{r1}}^{\theta_{r2}} A(\theta_r) d\theta_r = - \frac{\Delta \theta_r}{2} + \frac{1 - 2 \xi}{4 \xi \eta} \int_{\theta_{r1}}^{\theta_{r2}} \frac{d\theta_r}{a - \cos \theta_r}.
\]

For the sake of space, define the function,

\[
R(\theta_i) := \arctan \left( \frac{\tan \left( \frac{\theta_i}{2} \right)}{b} \right)
\]

IV. DERIVATION OF EQUATIONS OF MOTION FOR A TAILED SYSTEM

Equipped with the kinematic results of [1, Sec. II], the balance of angular momentum for a general tailed system about the COM of each body is (see Fig. 3),

\[
\dot{H}_b = \tau E_3 + (-l_b e_{r_b}) \times F_p,
\]

\[
\dot{H}_t = -\tau E_3 + (-l_t e_{r_t}) \times (-F_p),
\]

where \( \tau \) denotes the torque output of the power train, and \( F_p \) is the pin constraint force (see Fig. 3). Since both the body and the COM frame are subject to the same gravitational acceleration, the force of gravity does not appear in the pin force, which is simply \( F_p = m_b g_b \). The body acceleration relative to the COM is found by differentiating (49).

Making the substitutions, the integral in (55) simplifies to,

\[
\int \frac{d\theta_r}{a - \cos \theta_r} = \int \frac{1}{a - \frac{1 - t^2}{1 + t^2}} \frac{2}{1 + t^2} dt
\]

\[
= \int \frac{2}{a(1 + t^2) - (1 - t^2)} dt
\]

\[
= \int \frac{1}{a(1 - t^2) + (a + 1)t^2} dt
\]

\[
= \frac{2}{a + 1} \arctan \frac{t}{b} + C_1
\]

\[
= \frac{2}{b(a + 1)} \arctan \left( \frac{\tan \left( \frac{\theta_r}{2} \right)}{b} \right) + C_2.
\]
A. Nondimensionalization of nonlinear tail dynamics

The equations of motion [1, Eqn. 38] and [1, Eqn. 39] can be written in the generalized coordinates \((\theta_b, \theta_r)\) by substituting for \(\theta_t = \theta_b + \theta_r\) and applying the change of basis to [1, Eqn. 38],

\[
\begin{bmatrix}
\dot{\theta}_b \\
\dot{\theta}_r
\end{bmatrix}
= \begin{bmatrix}
m_r l_b l_t \sin \theta_r (2 \dot{\theta}_b \dot{\theta}_r + \dot{\theta}_b^2) \\
-m_r l_b l_t \sin \theta_r \dot{\theta}_b^2
\end{bmatrix}
\begin{bmatrix}
0 \\
-1
\end{bmatrix} \tau,
\]

(65)

with a mass matrix,

\[
M(\theta_r) = \begin{bmatrix}
I_b + m_r (l_t^2 + l_b^2 - 2 l_b l_t \cos \theta_r) & I_t + m_r l_t^2 - m_r l_b l_t \cos \theta_r \\
I_t + m_r l_t^2 - m_r l_b l_t \cos \theta_r & I_t + m_r l_t^2
\end{bmatrix}.
\]

Following the process of [1, Sec. II-B], we substitute the template motor model for the torque and the scaling factors from the template, [1, Eqn. 14], along with a new scaling for the relative angle, \(\theta'_r := \theta_r / \gamma\) (note that unlike for \(\theta_b\), we do not normalize for final position). Normalizing by \(\frac{\xi_t}{\xi_b} (I_b + m_r l_b^2)\), we define the dimensionless mass matrix,

\[
\tilde{M}(\theta_r) = \begin{bmatrix}
\frac{1-\xi_t}{\xi_b} + 2 \eta \cos \theta_r & 1 - \eta \cos \theta_r \\
1 - \eta \cos \theta_r & 1
\end{bmatrix},
\]

and the dimensionless Coriolis terms,

\[
\tilde{C}(\theta_r, \theta'_r, \phi'_r) = \eta \theta_b, f \sin \theta_r \left[ \frac{2 \ddot{\theta}_b'}{\pi_b, f} + \frac{(\ddot{\theta}_b')^2}{\pi_b, f} \right],
\]

(66)

resulting in dimensionless system dynamics,

\[
\tilde{M}(\theta_r) \begin{bmatrix}
\frac{\tilde{\theta}_b'}{\pi_b, f} \\
\frac{\tilde{\theta}_r'}{\pi_r, f}
\end{bmatrix} + \tilde{C}(\theta_r, \theta'_r, \phi'_r) = \begin{bmatrix}
0 \\
-1
\end{bmatrix} \left( \frac{1-\xi_t}{\xi_t} \right) \tilde{\tau},
\]

(67)

with

\[
\tilde{\tau} = \frac{1}{\hat{\omega}_m} \left( 1 - \frac{\xi_t}{\hat{\omega}_m} \right)
\]

(68)
during acceleration, and \(\hat{\tau} = 1/\hat{\omega}_m\) during braking.

V. Derivation of the connection for assemblage of limbs

Here we consider a simplified case, where all appendages are parallel (but potentially out of phase by \(180^\circ\), as in RHex’s alternating tripod gait), and the \(N\) limbs are arranged with pivots along the centerline of the robot’s body (along which the body’s COM also falls). Again, the limbs are driven by a high-gain synchronizing control such that all \(N\) legs share the same angle \(\theta_i\), modulo the phasing noted above.

Using the same reference frames from the tail case, Section III, above, let \(e_{rb}\) be the vector parallel to the body axis, and \(e_{rt}\) be the vector to which all limbs are parallel. Denote the vector from body COM to the \(i\)th pivot by,

\[
p_i := \ell_i e_{rb},
\]

(69)

and the vector from pivot to appendage COM by,

\[
t_i := s_i l_i e_{rt},
\]

(70)

where \(\ell_i\) is the position of the pivot along the body (\(\ell\) is negative for pivots behind the body COM), \(l_i\) is the length of the \(i\)th limb, and \(s_i := \pm 1\) is negative for legs out of phase with \(e_{rt}\) by \(\pi\). The vector from system COM to appendage COM is,

\[
r_i := r_b + p_i + t_i = r_b + \ell_i e_{rb} + s_i l_i e_{rt},
\]

(71)

and the relation between system COM and segment COMs is,

\[
m_{tot} \mathbf{r}_{com} = m_b r_b + \sum_{i=1}^{N} m_i r_i,
\]

(72)

where \(m_i\) is the mass of the \(i\)th appendage, and \(m_{tot} := m_b + \sum_{i=1}^{N} m_i\) is the total system mass. Placing the origin at the system COM \((\mathbf{r}_{com} = 0)\) and solving for \(r_b\),

\[
0 = m_b r_b + \sum_{i=1}^{N} m_i (r_b + \ell_i e_{rb} + s_i l_i e_{rt})
\]

(73)

\[
m_{tot} r_b = - \sum_{i=1}^{N} m_i (\ell_i e_{rb} + s_i l_i e_{rt})
\]

(74)

\[
r_b = - \frac{1}{m_{tot}} \left( e_{rb} \sum_{i=1}^{N} m_i \ell_i + e_{rt} \sum_{i=1}^{N} m_i s_i l_i \right).
\]

(75)

If \(\sum_{i=1}^{N} m_i \ell_i = 0\) (that is, the mass-weighted pivot distances from body COM are symmetric), then \(r_b\) is strictly parallel to \(e_{rt}\),

\[
r_b = c e_{rt}; \quad c := - \frac{1}{m_{tot}} \sum_{i=1}^{N} m_i s_i l_i,
\]

(76)

and the vector to the \(i\)th appendage COM simplifies to,

\[
r_i = \ell_i e_{rb} + (c + s_i l_i) e_{rt}.
\]

(77)

The connection can be derived from the total angular momentum; extending (39) to multiple appendages,

\[
\mathbf{H}_{O,t} = I_b \dot{\theta}_b \mathbf{E}_3 + r_b \times (m_b \dot{r}_b) + \sum_{i=1}^{N} \left( I_i \dot{\theta}_i \mathbf{E}_3 + r_i \times (m_i \dot{r}_i) \right).
\]

(78)
The moment of linear momentum due to the body mass can be simplified using (76),
\[ \mathbf{r}_b \times m_b \mathbf{\dot{r}}_b = c \mathbf{e}_r \times m_b c \mathbf{e}_s t = m_b c^2 \dot{\theta}_b \mathbf{E}_3. \]
The moment of linear momentum due to each appendage can be simplified using (77),
\[ \mathbf{r}_i \times m_i \mathbf{\dot{r}}_i = m_i \left( \ell_i \mathbf{e}_r + (c + s_i \ell_i) \mathbf{e}_s \right) \times \left( \ell_i \dot{\theta}_b \mathbf{e}_b + (c + s_i \ell_i) \dot{\theta}_t \mathbf{e}_s \right) = m_i \left( \ell_i^2 \dot{\theta}_b + \ell_i (c + s_i \ell_i) (\dot{\theta}_b + \dot{\theta}_t) \cos \theta_r + (c + s_i \ell_i)^2 \dot{\theta}_t \right) \mathbf{E}_3. \]

With these simplifications, the magnitude of the angular momentum, (78), in the \( \mathbf{E}_3 \) direction, \( H_{O,l} \mathbf{E}_3 := H_{O,l} \), is,
\[ H_{O,l} = I_b \dot{\theta}_b + m_b c^2 \dot{\theta}_t + \sum_{i=1}^{N} \left( I_i \dot{\theta}_b + m_i \left( \ell_i^2 \dot{\theta}_b + \ell_i (c + s_i \ell_i) (\dot{\theta}_b + \dot{\theta}_t) \cos \theta_r \right) \right), \]
where the only remaining configuration dependent term is,
\[ \sum_{i=1}^{N} m_i \ell_i (c + s_i \ell_i) (\dot{\theta}_b + \dot{\theta}_t) \cos \theta_r, \]
and hence one criterion for configuration independence is,
\[ \sum_{i=1}^{N} m_i \ell_i (c + s_i \ell_i) = 0. \] (79)

This is satisfied if all appendages have equal length, \( \ell_i \), and phase, \( s_i \), (as when all six of XRL’s legs share the same angle) and if \( \sum_{i=1}^{N} m_i \ell_i = 0 \) (as required for the simplification of \( \mathbf{r}_b \)). Note that if an assemblage of \( N \) appendages satisfy this condition, then the addition of an appendage with \( \ell_i = 0 \) will result in an assemblage of \( N + 1 \) appendages that will satisfy this condition as well.

For limb systems that satisfy (79), the magnitude of the angular momentum, (78), in the \( \mathbf{E}_3 \) direction simplifies to,
\[ H_{O,l} = \left( I_b + \sum_{i=1}^{N} m_i \ell_i^2 \right) \dot{\theta}_b + \left( m_b c^2 + \sum_{i=1}^{N} \left( I_i + m_i s_i \ell_i - \frac{\sum_{j=1}^{N} m_j s_j \ell_j}{m_{tot}} \right)^2 \right) \dot{\theta}_t. \] (80)

If, further, all legs have identical mass, length, and inertia, which we will call \( m_i \), \( \ell_i \) and \( I_i \) for comparison with the tail anchor, and the pivot locations are symmetric across the body centerline, i.e. \( \sum \ell_i = 0 \),
\[ H_{O,l} = (I_b + m_t \sum_{i=1}^{N} \ell_i^2) \dot{\theta}_b + N I_t \dot{\theta}_t + \]
\[ m_t \ell_t^2 \left( \frac{m_b m_t}{m_{tot}} \left( \sum_{i=1}^{N} s_i \right)^2 + \sum_{i=1}^{N} \left( s_i - \frac{m_t \sum_{j=1}^{N} s_j}{m_{tot}} \right)^2 \right) \dot{\theta}_t. \] (81)

To simplify further, assume first that \( \sum s_i = 0 \),
\[ H_{O,l} = (I_b + m_t \sum_{i=1}^{N} \ell_i^2) \dot{\theta}_b + N (I_t + m_t \ell_t^2) \dot{\theta}_t, \] (82)
which, after a change of coordinates to \( (\theta_b, \theta_r) \), is as claimed in [1, Eqn. 51]. If instead \( \sum s_i = N \),
\[ H_{O,l} = (I_b + m_t \sum_{i=1}^{N} \ell_i^2) \dot{\theta}_b + N I_t \dot{\theta}_t + \]
\[ m_t \ell_t^2 \left( \frac{m_b m_t}{m_{tot}} N^2 + \sum_{i=1}^{N} \left( m_b + N m_t - m_t N \right)^2 \right) \dot{\theta}_t, \] (83)
\[ = (I_b + m_t \sum_{i=1}^{N} \ell_i^2) \dot{\theta}_b + N I_t \dot{\theta}_t + \]
\[ m_t \ell_t^2 \left( \frac{m_b m_t}{m_{tot}} N (N m_t + m_b) \right) \dot{\theta}_t, \] (84)
which, after a change of coordinates to \( (\theta_b, \theta_r) \) and substituting the definition of \( m_t r_t \), is as claimed in [1, Eqn. 52].

REFERENCES